Quantum correlations and least disturbing local measurements

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We examine the evaluation of the minimum information loss due to an unread local measurement in mixed states of bipartite systems, for a general entropic form. Such quantity provides a measure of quantum correlations, reducing for pure states to the generalized entanglement entropy, while in the case of mixed states it vanishes just for classically correlated states with respect to the measured system, as the quantum discord. General stationary conditions are provided, together with their explicit form for general two-qubit states. Closed expressions for the minimum information loss as measured by quadratic and cubic entropies are also derived for general states of two-qubit systems. As application, we analyze the case of states with maximally mixed marginals, where a general evaluation is provided, as well as X states and the mixture of two aligned states.

PACS numbers: 03.67.-a, 03.65.Ud, 03.65.Ta

I. INTRODUCTION

There is currently a great interest on new measures of quantum correlations for mixed states, different from the entanglement measures [1]. Quantum entanglement is essential for quantum teleportation [2, 3] and also for pure state quantum computation, where its increase with system size is necessary to achieve an exponential speedup over classical computation [4, 5]. However, the computation model proposed by Knill and Laflamme [6] has shown that for mixed states, such speedup can in principle be achieved without entanglement [7]. This suggests the subsistence of useful quantum correlations in some separable mixed states, which, we recall, are defined as convex mixtures of product states [8]. While a separable pure state is a product state, separable mixed states comprise product states, mixtures of commuting products and also mixtures of non-commuting product states. The latter can possess entangled eigenstates and give rise to non-classical capabilities.

Consequently, measures such as the quantum discord [9–12] have recently received much attention. While coinciding with the entanglement entropy for pure states, the quantum discord is non-zero for separable mixed states of the last type, vanishing just for classically or one-way classically correlated states, i.e., states diagonal in a standard or conditional product basis. The circuit of [6] was in fact shown in [13] to exhibit a non-negligible discord. Other measures with similar properties include the one-way information deficit [14, 15], the geometric discord [16], based on the standard Hilbert-Schmidt norm, and the general entropic measures which we defined in [17], based on generalized entropic forms. The latter contain the two previous measures as particular cases, embedding them in a unified picture. Since they are applicable with entropic forms complying with minimum requirements, they offer, like the geometric discord, the possibility of easier evaluations, allowing at the same time to identify some universal features exhibited by all these measures [17]. Related generalized measures vanishing just for full classically correlated states, like those of [18] and [19], were also considered [17]. Let us remark that important quantum capabilities of separable states with non-zero discord, and hence non-zero values of the previous measures, were recently unveiled [15, 20–23]. Other relevant properties of the quantum discord and its evaluation in specific states and scenarios were discussed in [24–35].

The aim of this work is to analyze the explicit evaluation of the generalized measures of [17] in some important general cases. We first provide in Sec. II the general stationary condition that the least disturbing local measurement should satisfy, including conditions for its independence from the entropy employed (universality), together with its explicit form for general two-qubit states. Here we show that in addition to the quadratic case (geometric discord), the measure based on a cubic function of the density matrix ("cubic" discord) can also be exactly evaluated for any state of two qubits. Moreover, for two-qubit states this measure shares with the geometric discord the same pure state limit, where they are both proportional to the square of the concurrence [36, 37]. As specific applications, we provide in sec. III the general expression for two-qubit states with maximally mixed reduced states, valid for any entropic form, analyzing its main features. We also examine their evaluation in the so-called X states [32], where explicit expressions for the quadratic and cubic cases are provided, and for the important case of a mixture of two aligned states [33], which represents in particular the exact state of a pair in the ground state of a finite XY ferromagnetic spin 1/2 chain in the vicinity of the factorizing field [38]. Differences with the quantum discord, related in particular with the minimizing measurement, are also discussed. Conclusions are finally drawn in Sec. IV.

II. FORMALISM

A. Information loss by unread local measurement

Let us consider a bipartite system $A+B$ initially in a state ρ_{AB}. After an unread local von Neumann measurement in system B, defined by orthogonal one dimensional projectors $P_j^B = I \otimes P_j$, with $P_j = |j_B\rangle\langle j_B|$ ($\sum_j P_j = I$, $P_j P_{j'} = \delta_{jj'} P_j$), the joint state becomes

$$\rho'_{AB} = \sum_j P_j^B \rho_{AB} P_j^B = \sum_j p_j \rho_{AB/j}\,, \tag{1}$$

where $p_j = \mathrm{Tr}\, \rho_{AB} P_j^B$ is the probability of outcome j and $\rho_{AB/j} = P_j^B \rho_{AB} P_j^B / p_j$ the state after such outcome. The state (1) is just the diagonal of ρ_{AB} in a conditional product basis formed by the states $|i_j j\rangle \equiv |i_{jA}\rangle|j_B\rangle$, with $|i_{jA}\rangle$ the eigenstates of $\rho_{A/j} = \mathrm{Tr}_B \rho_{AB/j}$. The loss of information due to such measurement, i.e., the information contained in the off-diagonal elements of the original ρ_{AB} in the previous basis, can be quantified by the quantity [17]

$$I_f^{M_B}(\rho_{AB}) = S_f(\rho'_{AB}) - S_f(\rho_{AB})\,, \tag{2}$$

where $S_f(\rho)$ denotes a generalized entropy of the form

$$S_f(\rho) = \mathrm{Tr}\, f(\rho)\,, \tag{3}$$

with $f : [0,1] \to \Re$ a smooth strictly concave function ($f''(p) < 0$ for $p \in (0,1)$) satisfying $f(0) = f(1) = 0$ [39, 40]. This ensures $S_f(\rho) \geq 0$ for any state ρ, with $S_f(\rho) = 0$ if and only if ρ is a pure state ($\rho^2 = \rho$), and $S_f(\rho)$ maximum, at fixed dimension n, for the maximally mixed state $\rho = I/n$. Eq. (2) is then *non-negative* for any S_f of the previous form, vanishing only if the original ρ_{AB} remains unchanged by such measurement. This positivity follows from the majorization relation [3, 40, 41] $\rho'_{AB} \prec \rho_{AB}$ (ρ'_{AB} more mixed than ρ_{AB}) satisfied by the post-measurement state, which implies $S_f(\rho'_{AB}) \geq S_f(\rho_{AB})$ for all such S_f [17]. Moreover, the previous entropic inequality implies in fact majorization when valid for *all* S_f of the previous form [42].

The minimum of $I_f^{\rm M_B}$ among all local measurements,

$$I_f^B(\rho_{AB}) = \underset{M_B}{\mathrm{Min}}\, I_f^{M_B}(\rho_{AB})\,, \tag{4}$$

provides a measure of the quantum correlations between A and B present in the original state and destroyed by local measurement [17]. It vanishes only if ρ_{AB} is already of the "classical" (with respect to B) form (1). For such states there is an unread local measurement in B (M_B) which leaves the state invariant. Eq. (4) is obviously not affected by local unitary transformations.

In the case of pure states ($\rho_{AB}^2 = \rho_{AB}$), it can be shown that (4) becomes the generalized entanglement entropy

$$I_f^B(\rho_{AB}) = E_f(A,B) \equiv S_f(\rho_A) = S_f(\rho_B), \tag{5}$$

where $\rho_A = \mathrm{Tr}_B\, \rho_{AB}$ and ρ_B are the reduced states of each subsystem [17]. Hence, pure state entanglement can be seen as the minimum information loss due to a local measurement. In this case $I_f^B(\rho_{AB}) = I_f^A(\rho_{AB})$, an identity which does not hold in general for mixed states.

In the von Neumann case $S_f(\rho) = S(\rho) \equiv -\mathrm{Tr}\rho \log \rho$, Eq. (2) can be also written as [17]

$$I^{M_B}(\rho_{AB}) = S(\rho'_{AB}) - S(\rho_{AB}) = S(\rho_{AB}||\rho'_{AB})\,, \tag{6}$$

where $S(\rho||\rho') = -\mathrm{Tr}\, \rho(\log \rho' - \log \rho)$ is the *relative* entropy [3, 40, 43] (a non-negative quantity), since ρ'_{AB} is the diagonal of ρ_{AB} in a certain basis. The minimum I^B of Eq. (6) coincides with the one-way information deficit [14, 15] and also with one of the measures discussed in [18]. In the case of pure states, I^B reduces to the standard entanglement entropy $E(A,B) = S(\rho_A) = S(\rho_B)$.

In the case of the so-called linear entropy

$$S_2(\rho) = 2(1 - \mathrm{Tr}\, \rho^2)\,, \tag{7}$$

which is a quadratic function of ρ and corresponds to $f(\rho) = 2\rho(1-\rho)$ in (3), Eq. (2) can be written as [17]

$$I_2^{M_B}(\rho_{AB}) = S_2(\rho'_{AB}) - S_2(\rho_{AB}) = 2||\rho'_{AB} - \rho_{AB}||^2\,, \tag{8}$$

where $||O||^2 = \mathrm{Tr}\, O^\dagger O$ is the squared Hilbert-Schmidt norm. The ensuing minimum (4), to be denoted here as I_2^B, becomes then equivalent [17] to the geometric discord of ref. [16], defined as the minimum Hilbert-Schmidt distance between ρ_{AB} and any classically correlated state of the form (1). In the case of pure states, I_2^B reduces to the square of the pure state concurrence (i.e., the tangle), defined as $C_{AB} = \sqrt{2(1 - \mathrm{Tr}\, \rho_A^2)}$ [37].

In the same way we may define the q information loss

$$I_q^{M_B}(\rho_{AB}) = S_q(\rho'_{AB}) - S_q(\rho_{AB})\,, \tag{9}$$

$$S_q(\rho) = (1 - \mathrm{Tr}\rho^q)/(1 - 2^{1-q})\,,\; q > 0\,, \tag{10}$$

where $S_q(\rho)$ is the so-called Tsallis entropy [44], which corresponds to $f(\rho) = (\rho - \rho^q)/(1 - 2^{1-q})$ in (3) and is a function of the Renyi entropy. Eq. (10) reduces to the linear entropy (7) for $q = 2$ and to the von Neumann entropy for $q \to 1$, with $\log = \log_2$ for the present normalization (chosen such that $S_q(\rho) = 1$ for a maximally mixed single qubit state, i.e., $2f(1/2) = 1$). Eq. (9) allows in particular to switch continuously from the von Neumann case (6) to the quadratic case (7).

On the other hand, the original quantum discord [9–12] is based on the von Neumann entropy and can be written (considering von Neumann measurements) as

$$D^B(\rho_{AB}) = \underset{M_B}{\mathrm{Min}}[I^{M_B}(\rho_{AB}) - I^{M_B}(\rho_B)]\,. \tag{11}$$

It contains an additional term $I^{M_B}(\rho_B) = S(\rho'_B) - S(\rho_B)$ related to the local information loss and was actually defined in [9] as the minimum difference between the initial mutual information

$$I(A:B) = S(\rho_A \otimes \rho_B) - S(\rho_{AB})\,, \tag{12}$$

where $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$, and that after the local measurement, $I^{M_B}(A:B) = S(\rho'_A) + S(\rho'_B) - S(\rho'_{AB})$. Since $\rho'_A = \rho_A$, such difference reduces to Eq. (11).

The information loss (2) can be regarded in fact as a type of generalized mutual information. Eq. (12) is a measure of the total correlations between A and B in the original state, absent in the product state $\rho_A \otimes \rho_B$. The latter is the state which *maximizes* the von Neumann entropy subject to the constraint of providing just all *local* averages $\langle O\otimes I\rangle$ and $\langle I\otimes O\rangle$, i.e., the correct reduced states ρ_A and ρ_B. This is in fact what is expressed by the positivity of Eq. (12): Any other state ρ_{AB} with the same local reduced states has a smaller entropy.

On the other hand, the post-measurement state (1) can be seen as the *more mixed* state providing the same averages as ρ_{AB} of all observables of the form $\sum_j \alpha_j O_j\otimes P_j$, diagonal in the local basis defined by M_B (as $\mathrm{Tr}\,\rho_{AB}\,O\otimes P_j=\mathrm{Tr}\,\rho'_{AB}\,O\otimes P_j$), such that $S_f(\rho'_{AB})\geq S_f(\rho_{AB})$ $\forall$ S_f. The difference $I_f^{M_B}$ is then a measure of the correlations $\langle O\otimes |j_B\rangle\langle k_B|\rangle$, $k\neq j$, contained in the original state ρ_{AB} and absent in ρ'_{AB}. In particular, if M_B is a measurement in a basis where ρ_B is diagonal, ρ'_{AB} reproduces not only ρ_A ($\rho'_A=\mathrm{Tr}_B\,\rho'_{AB}=\rho_A$ $\forall$ M_B) but also ρ_B ($\rho'_B=\mathrm{Tr}_A\,\rho'_{AB}=\rho_B$ for this measurement), as well as all averages $\langle O\otimes P_j\rangle$, being the more mixed state with such property. Notice that in contrast with ρ'_{AB}, the state $\rho_A\otimes\rho_B$ is in general not more mixed than the original state ($\rho_A\otimes\rho_B\not\prec\rho_{AB}$), so that the positivity of Eq. (12) cannot be extended to a general entropy.

B. General stationary condition

Let us now derive the equations determining the least disturbing local measurement defined by Eq. (4).

Theorem 1. For a given entropic function f, the least disturbing local measurement satisfies the equation

$$\mathrm{Tr}_A[f'(\rho'_{AB}),\rho_{AB}]=0\,, \tag{13}$$

where f' is the derivative of f and ρ'_{AB} the post-measurement state (1).

Proof: The generalized entropy of the state (1) is

$$S_f(\rho'_{AB})=\sum_{i,j}f(p_j^i)\,,\;\;p_j^i=\langle i_jj|\rho_{AB}|i_jj\rangle\,, \tag{14}$$

where $\langle i_jj|\rho_{AB}|k_jj\rangle=\delta_{ik}p_j^i$. Considering a small unitary variation of the local measurement basis, such that $\delta|j_B\rangle=(e^{i\delta h}-1)|j_B\rangle\approx i\delta h|j_B\rangle$, with δh a small local hermitian operator, we have $\delta p_j^i\approx i\langle i_jj|[\rho_{AB},\delta h_B]|i_jj\rangle$ up to first order in δh, with $\delta h_B=I\otimes\delta h$. Hence,

$$\begin{aligned}\delta I_f^{M_B}&=\sum_{i,j}f'(p_j^i)\delta p_j^i=i\mathrm{Tr}\,[f'(\rho'_{AB}),\rho_{AB}]\delta h_B\\&=i\mathrm{Tr}_B\,(\mathrm{Tr}_A[f'(\rho'_{AB}),\rho_{AB}])\delta h\,.\end{aligned}$$

The condition $\delta I_f^{M_B}=0$ $\forall$ δh leads then to Eq. (13).

Eq. (13) implies explicitly $\sum_i f'(p_j^i)\langle i_jj|\rho_{AB}|i_jk\rangle=\sum_i f'(p_k^i)\langle i_kj|\rho_{AB}|i_kk\rangle$ $\forall$ k,j, and determines a certain set of feasible local basis $\{|j_B\rangle\}$. Note that the states $|i_j\rangle$ of A depend in general on j.

The minimizing local basis $\{|j_B\rangle\}$ will not diagonalize, in general, the reduced state ρ_B. Nonetheless, Eq. (13) entails that the local eigenstates can be optimum in some important situations: If in a standard product basis $\{|ij\rangle=|i_A\rangle|j_B\rangle\}$ formed by eigenstates of ρ_A and ρ_B the only off-diagonal elements of ρ_{AB} are $\langle ij|\rho_{AB}|kl\rangle$ with $i\neq k$ *and* $j\neq l$, such that

$$\langle ij|\rho_{AB}|ik\rangle=\delta_{jk}p_j^i\,,\;\;\langle ij|\rho_{AB}|lj\rangle=\delta_{il}p_j^i\,, \tag{15}$$

Eq. (13) is trivially satisfied $\forall$ S_f for a measurement in the basis $\{|j_B\rangle\}$. Such basis would then provide *a universal stationary point of* I_f^B. This is precisely the case of a pure state, written in the Schmidt basis as $|\Psi_{AB}\rangle=\sum_k\sqrt{p_k}|k_Ak_B\rangle$, and also of a mixture of $|\Psi_{AB}\rangle$ with the maximally mixed state,

$$\rho_{AB}=x|\Psi_{AB}\rangle\langle\Psi_{AB}|+\tfrac{1-x}{n}I\,,\;x\in[0,1]\,,$$

where Eqs. (15) and hence (13) will be satisfied $\forall$ f for a measurement in the basis $\{|k_B\rangle\}$. It was shown in [17] that such basis provides *the universal least disturbing local measurement* for these states, minimizing $I_f^{M_B}$ $\forall$ S_f.

In the case of the linear entropy, $f'(\rho'_{AB})\propto I-2\rho'_{AB}$ and Eq. (13) becomes just $\mathrm{Tr}_A[\rho'_{AB},\rho_{AB}]=0$, indicating that the post-measurement state ρ'_{AB} should locally (in B) commute with the original state.

In the case of the original discord (11), the additional local term leads in the variation to the modified equation

$$\mathrm{Tr}_A[f'(\rho'_{AB}),\rho_{AB}]-[f'(\rho'_B),\rho_B]=0\,, \tag{16}$$

where here $f'(\rho)$ can be replaced by $-\log\rho$.

C. The two-qubit case

Let us now examine in detail the case of two-qubits. Any state of a two qubit system can be written as

$$\rho_{AB}=\frac{1}{4}(I+\boldsymbol{r}_A\cdot\boldsymbol{\sigma}_A+\boldsymbol{r}_B\cdot\boldsymbol{\sigma}_B+\boldsymbol{\sigma}_A^tJ\boldsymbol{\sigma}_B)\,, \tag{17}$$

where $\boldsymbol{\sigma}_A\equiv\boldsymbol{\sigma}\otimes I$, $\boldsymbol{\sigma}_B\equiv I\otimes\boldsymbol{\sigma}$, with $\boldsymbol{\sigma}^t=(\sigma_x,\sigma_y,\sigma_z)$ the Pauli operators and I the identity (in the corresponding space). The basic traces $\mathrm{tr}\,\sigma_\mu=0$, $\mathrm{tr}\,\sigma_\mu\sigma_\nu=2\delta_{\mu\nu}$ for $\mu,\nu=x,y,z$, ensure that

$$\boldsymbol{r}_A=\langle\boldsymbol{\sigma}_A\rangle\,,\;\boldsymbol{r}_B=\langle\boldsymbol{\sigma}_B\rangle\,,\;J=\langle\boldsymbol{\sigma}_A\boldsymbol{\sigma}_B^t\rangle\,,$$

i.e., $J_{\mu\nu}=\langle\sigma_{A\mu}\,\sigma_{B\nu}\rangle$, where $\langle O\rangle=\mathrm{Tr}\,\rho_{AB}\,O$.

Any complete local projective measurement in B can be considered as a spin measurement along the direction of a unit vector $\boldsymbol{k}$, represented by the orthogonal projectors $P_{\pm\boldsymbol{k}}=\frac{1}{2}(I\pm\boldsymbol{k}\cdot\boldsymbol{\sigma})$. This leaves just those elements of ρ_{AB} proportional to $\boldsymbol{k}\cdot\boldsymbol{\sigma}$, leading to the post-measurement state

$$\rho'_{AB}=\frac{1}{4}[I+\boldsymbol{r}_A\cdot\boldsymbol{\sigma}_A+(\boldsymbol{r}_B\cdot\boldsymbol{k})\boldsymbol{k}\cdot\boldsymbol{\sigma}_B+(\boldsymbol{\sigma}_A^tJ\boldsymbol{k})(\boldsymbol{k}\cdot\boldsymbol{\sigma}_B)]\,, \tag{18}$$

which corresponds to $\boldsymbol{r}_B\rightarrow\boldsymbol{k}\boldsymbol{k}^t\boldsymbol{r}_B$ and $J\rightarrow J\boldsymbol{k}\boldsymbol{k}^t$ in (17). The information loss due to this measurement will be denoted as $I_f^{\boldsymbol{k}}\equiv S_f(\rho'_{AB})-S_f(\rho_{AB})$.

We now show that *the general stationary condition for the measurement direction* $\boldsymbol{k}$ *in* B *reads*

$$\alpha_1 \boldsymbol{r}_B + \alpha_2 J^t \boldsymbol{r}_A + \alpha_3 J^t J \boldsymbol{k} = \lambda \boldsymbol{k}\,, \tag{19}$$

i.e., $\boldsymbol{k} \times (\alpha_1 \boldsymbol{r}_B + \alpha_2 J^t \boldsymbol{r}_A + \alpha_3 J^t J \boldsymbol{k}) = \boldsymbol{0}$, where λ is a proportionality factor and the coefficients α_i are given by

$$(\alpha_1, \alpha_2, \alpha_3) = \tfrac{1}{4} \sum_{\mu,\nu=\pm 1} f'(p_\nu^\mu)(\nu, \tfrac{\nu\mu}{|\boldsymbol{r}_A + \nu J \boldsymbol{k}|}, \tfrac{\mu}{|\boldsymbol{r}_A + \nu J \boldsymbol{k}|})\,, \tag{20}$$

with p_ν^μ $(\mu, \nu = \pm 1)$ the eigenvalues of (18):

$$p_\nu^\mu = \tfrac{1}{4}(1 + \nu \boldsymbol{r}_B \cdot \boldsymbol{k} + \mu |\boldsymbol{r}_A + \nu J \boldsymbol{k}|)\,. \tag{21}$$

Proof: The state (18) is diagonal in the conditional product basis formed by the eigenstates of $\boldsymbol{k} \cdot \boldsymbol{\sigma}_B$ and $(\boldsymbol{r}_A + \nu J \boldsymbol{k}) \cdot \boldsymbol{\sigma}_A$, with $\nu = \pm 1$ the eigenvalues of $\boldsymbol{k} \cdot \boldsymbol{\sigma}_B$, which leads to the eigenvalues (21). We can then write

$$f'(\rho'_{AB}) = \tfrac{1}{4} \sum_{\nu,\mu} f'(p_\nu^\mu)(I + \mu \tfrac{\boldsymbol{r}_A + \nu J \boldsymbol{k}}{|\boldsymbol{r}_A + \nu J \boldsymbol{k}|} \cdot \boldsymbol{\sigma}_A)(I + \nu \boldsymbol{k} \cdot \boldsymbol{\sigma}_B)\,.$$

Using now the basic trace relations and $[\boldsymbol{r} \cdot \boldsymbol{\sigma}, \boldsymbol{s} \cdot \boldsymbol{\sigma}] = 2i(\boldsymbol{r} \times \boldsymbol{s}) \cdot \boldsymbol{\sigma}$, we obtain $\mathrm{Tr}_A\,[(\boldsymbol{r} \cdot \boldsymbol{\sigma}_A)(\boldsymbol{s} \cdot \boldsymbol{\sigma}_B), \boldsymbol{\sigma}_A^t J \boldsymbol{\sigma}_B] = 4i(\boldsymbol{s} \times J^t \boldsymbol{r}) \cdot \boldsymbol{\sigma}_B$ and hence

$$\mathrm{Tr}_A\,[f'(\rho'_{AB}), \rho_{AB}] = i[\boldsymbol{k} \times (\alpha_1 \boldsymbol{r}_B + \alpha_2 J^t \boldsymbol{r}_A + \alpha_3 J^t J \boldsymbol{k})] \cdot \boldsymbol{\sigma}_B\,,$$

with α_i given by (20). Eq. (13) leads then to Eq. (19).

We can also check Eq. (19) directly. From (21), we have $\delta p_\nu^\mu = \frac{\nu}{4}(\boldsymbol{r}_B + \mu \frac{J^t(\boldsymbol{r}_A + \nu J \boldsymbol{k})}{|\boldsymbol{r}_A + \nu J \boldsymbol{k}|}) \cdot \delta \boldsymbol{k}$ for changes $\delta \boldsymbol{k}$ in the direction of the local measurement apparatus, with $\boldsymbol{k} \cdot \delta \boldsymbol{k} = 0$ since $\boldsymbol{k}$ is a unit vector. The condition $\delta I_f^{\boldsymbol{k}} = \sum_{\nu,\mu} f'(p_\nu^\mu) \delta p_\nu^\mu = 0$ then implies $(\alpha_1 \boldsymbol{r}_B + \alpha_2 J^t \boldsymbol{r}_A + \alpha_3 J^t J \boldsymbol{k}) \cdot \delta \boldsymbol{k} = 0$, which leads to Eq. (19) since $\delta \boldsymbol{k}$ is orthogonal to $\boldsymbol{k}$.

Writing $\boldsymbol{k} = (\sin\gamma \cos\phi, \sin\gamma \sin\phi, \cos\gamma)$, Eq. (19) leads to a transcendental system for γ, ϕ ($\tan\gamma = d_z / \sqrt{d_x^2 + d_y^2}$, $\tan\phi = d_y / d_x$, with $\boldsymbol{d}$ the l.h.s. of (19)). Eq. (19) can be also seen as a self-consistent eigenvalue equation for the matrix $(\alpha_1 \boldsymbol{r}_B + \alpha_2 J^t \boldsymbol{r}_A) \boldsymbol{k}^t + \alpha_3 J^t J$.

Let us remark that the initial reduced local state $\rho_B = \mathrm{Tr}_A\, \rho_{AB} = \frac{1}{2}(I + \boldsymbol{r}_B \cdot \boldsymbol{\sigma})$, becomes

$$\rho'_B = \tfrac{1}{2}[I + (\boldsymbol{r}_B \cdot \boldsymbol{k})(\boldsymbol{k} \cdot \boldsymbol{\sigma})]\,, \tag{22}$$

after the local measurement. The minimizing direction $\boldsymbol{k}$ will depend on the matrix J and may obviously deviate from $\boldsymbol{r}_B$, changing the local state. A "transition" in the direction of the least disturbing $\boldsymbol{k}$, from $\boldsymbol{r}_B$ to the direction of the main eigenvector of $J^t J$, can then be expected from (19) as J increases from 0, whose details will in general depend on the choice of entropy (see sec. III).

In the case of the original quantum discord (11), the extra local contribution in (16) leads to the modified stationary condition (see also [34])

$$(\alpha_1 - \eta) \boldsymbol{r}_B + \alpha_2 J^t \boldsymbol{r}_A + \alpha_3 J^t J \boldsymbol{k} = \lambda \boldsymbol{k}\,, \tag{23}$$

where $\eta = \frac{1}{2} \sum_{\nu=\pm} \nu f'(p_\nu) = \frac{1}{2} \log(p_-/p_+)$, with $p_\nu = \sum_\mu p_\nu^\mu = \frac{1}{2}(1 + \nu \boldsymbol{r}_B \cdot \boldsymbol{k})$ the eigenvalues of ρ'_B. The extra term $-\eta \boldsymbol{r}_B$ will tend to diminish the effect of $\boldsymbol{r}_B$, favoring the direction determined by $J^t J$.

D. The quadratic and cubic information measures

While the evaluation of a general entropy $S_f(\rho)$ requires the determination of the eigenvalues of ρ, for those choices of f involving just low integer powers of ρ, $S_f(\rho)$ can be determined without their explicit knowledge. For instance, using just the basic trace relations $\mathrm{tr}\,\sigma_\mu = 0$. $\mathrm{tr}\,\sigma_\mu \sigma_\nu = 2\delta_{\mu\nu}$, the linear entropy (7) of any two qubit state can be evaluated as

$$S_2(\rho_{AB}) = \tfrac{3}{2} - \tfrac{1}{2}(|\boldsymbol{r}_A|^2 + |\boldsymbol{r}_B|^2 + ||J||^2)\,, \tag{24}$$

where $||J||^2 = \mathrm{tr}\, J^t J$ and $|\boldsymbol{r}|^2 = \boldsymbol{r} \cdot \boldsymbol{r} = \boldsymbol{r}^t \boldsymbol{r}$. For the post-measurement state (18), Eq. (24) becomes

$$S_2(\rho'_{AB}) = \tfrac{3}{2} - \tfrac{1}{2}|\boldsymbol{r}_A|^2 - \tfrac{1}{2}\boldsymbol{k}^t M_2 \boldsymbol{k}, \tag{25}$$

$$M_2 = \boldsymbol{r}_B \boldsymbol{r}_B^t + J^t J\,, \tag{26}$$

where M_2 is a positive semidefinite symmetric matrix.

The information loss becomes therefore

$$I_2^{\boldsymbol{k}} = \tfrac{1}{2}(|\boldsymbol{r}_B|^2 + ||J||^2 - \boldsymbol{k}^t M_2 \boldsymbol{k}) = \tfrac{1}{2}(\mathrm{tr}\, M_2 - \boldsymbol{k}^t M_2 \boldsymbol{k})\,. \tag{27}$$

The minimum $I_2^{\boldsymbol{k}}$ is just twice the *geometric discord*, defined and evaluated for two qubits in [16]. It corresponds then to $\boldsymbol{k}$ directed along the eigenvector with the *largest* eigenvalue of the matrix M_2:

$$I_2^B(\rho_{AB}) = \underset{\boldsymbol{k}}{\mathrm{Min}}\; I_2^{\boldsymbol{k}} = \tfrac{1}{2}(\mathrm{tr}\, M_2 - \lambda_1) = \tfrac{1}{2}(\lambda_2 + \lambda_3) \tag{28}$$

where $(\lambda_1, \lambda_2, \lambda_3)$ are the eigenvalues of M_2 sorted in *decreasing* order. A state ρ_{AB} which is already of the form (18) leads to $I_f^B(\rho_{AB}) = 0\ \forall\ S_f$ and is then characterized by a matrix M_2 of rank 1 (such that $\lambda_2 = \lambda_3 = 0$). It is verified that for $f'(p_\nu^\mu) \propto 1 - 2p_\nu^\mu$, Eq. (19) reduces to the present eigenvalue equation $M_2 \boldsymbol{k} = \lambda \boldsymbol{k}$, since $(\alpha_1, \alpha_2, \alpha_3) \propto (\boldsymbol{r}_B \cdot \boldsymbol{k}, 0, 1)$.

Another entropy which can be easily evaluated for any state of two qubits is the $q = 3$ case in (10),

$$S_3(\rho) = \tfrac{4}{3}(1 - \mathrm{Tr}\, \rho^3)\,. \tag{29}$$

Theorem 2. The entropy (29) of the general two qubit state (17), and the ensuing minimum information loss $I_3^B(\rho_{AB})$ due to a local measurement in B, are given by

$$S_3(\rho_{AB}) = \tfrac{1}{2}[S_2(\rho_{AB}) + 1 - (\boldsymbol{r}_A^t\, J\, \boldsymbol{r}_B - \det J)]\,, \tag{30}$$

$$I_3^B(\rho_{AB}) = \underset{\boldsymbol{k}}{\mathrm{Min}}\; I_3^{\boldsymbol{k}} = \tfrac{1}{4}(\mathrm{tr}\, M_3 - 2\det J - \lambda_1) = \tfrac{1}{4}(\lambda_2 + \lambda_3) - \tfrac{1}{2}\det J\,, \tag{31}$$

where $S_2(\rho_{AB})$ is the entropy (24) and $(\lambda_1, \lambda_2, \lambda_3)$ are the eigenvalues, sorted in decreasing order, of the matrix

$$M_3 = \boldsymbol{r}_B \boldsymbol{r}_B^t + J^t J + \boldsymbol{r}_B \boldsymbol{r}_A^t J + J^t \boldsymbol{r}_A \boldsymbol{r}_B^t\,, \tag{32}$$

which is positive semidefinite.

Proof: Applying the basic trace relations together with $\mathrm{tr}\,\sigma_\mu \sigma_\nu \sigma_\tau = 2i\epsilon_{\mu\nu\tau}$, with ϵ the full antisymmetric tensor $(\mu, \nu, \tau \in \{x, y, z\})$, the only terms with non-zero trace in ρ^3 are $\mathrm{Tr}(\boldsymbol{r}_A^t\sigma_A)(\boldsymbol{\sigma}_A^t J\boldsymbol{\sigma_B})(\boldsymbol{r}_B^t\boldsymbol{\sigma}_B) = 4\boldsymbol{r}_A^t J\boldsymbol{r}_B$ (and the same for its 3! permutations), $\mathrm{Tr}(\boldsymbol{\sigma}_A^t J\boldsymbol{\sigma}_B)^3 = 3!(2i)^2\det J$ and the quadratic terms appearing already in $\mathrm{Tr}\,\rho^2$. This leads to to Eq. (30).

Using Eq. (30), the cubic entropy of the post-measurement state (18) can be expressed as

$$S_3(\rho'_{AB}) = \tfrac{5}{4} - \tfrac{1}{4}(|\boldsymbol{r}_A|^2 + \boldsymbol{k}^t M_3\boldsymbol{k})\,, \qquad (33)$$

where M_3 is the matrix (32), since $\boldsymbol{r}_A^t J\boldsymbol{r}_B = \mathrm{tr}\,\boldsymbol{r}_B\boldsymbol{r}_A^t J = \mathrm{tr}\,J^t\boldsymbol{r}_A\boldsymbol{r}_B^t$ and $\det(J\boldsymbol{k}\boldsymbol{k}^t) = 0$. The matrix M_3 is clearly symmetric and also positive semi-definite, as $\boldsymbol{k}^t M_3\boldsymbol{k} \geq (|\boldsymbol{k}\cdot\boldsymbol{r}_B| - |J\boldsymbol{k}|)^2 \geq 0\ \forall\ \boldsymbol{k}$ if $|\boldsymbol{r}_A| \leq 1$. The information loss $I_3^{\boldsymbol{k}} = S_3(\rho'_{AB}) - S_3(\rho_{AB})$ is therefore

$$I_3^{\boldsymbol{k}} = \tfrac{1}{4}(\mathrm{tr}\,M_3 - 2\det J - \boldsymbol{k}^t M_3\boldsymbol{k})\,, \qquad (34)$$

where $\mathrm{tr}\,M_3 = |\boldsymbol{r}_B|^2 + ||J||^2 + 2\boldsymbol{r}_A^t J\boldsymbol{r}_B$. Its minimum corresponds then to $\boldsymbol{k}$ along the eigenvector with the *largest eigenvalue of* M_3, which leads to Eq. (31).

It is also verified that Eq. (19) leads in the present case to the same eigenvalue equation $M_3\boldsymbol{k} = \lambda\boldsymbol{k}$, since $(\alpha_1, \alpha_2, \alpha_3) \propto (\boldsymbol{r}_B^t\boldsymbol{k} + \boldsymbol{r}_A^t J\boldsymbol{k}, \boldsymbol{r}_B^t\boldsymbol{k}, 1)$ for $f'(p_\nu^\mu) \propto 1 - 3(p_\nu^\mu)^2$. As opposed to $I_2^{\boldsymbol{k}}$, the minimizing measurement can now depend also on $\boldsymbol{r}_A$ through the last terms of M_3. A state of the form (18) is then characterized by matrices M_3 and J of rank 1, such that Eq. (31) vanishes.

Let us notice that under arbitrary local rotations $\boldsymbol{\sigma}_\alpha \to R_\alpha\boldsymbol{\sigma}_\alpha$ for $\alpha = A, B$ ($R_\alpha R_\alpha^t = I$, $\det R_\alpha = +1$), we have $\boldsymbol{r}_\alpha \to R_\alpha^t\boldsymbol{r}_\alpha$ and $J \to R_A^t J R_B$ in (17), such that $M_2 \to R_B^t M_2 R_B$ and $M_3 \to R_B^t M_3 R_B$. Their eigenvalues remain therefore invariant. Of course, $\det J$ and all other terms in Eqs. (24) and (30) remain also unaltered.

Eqs. (24) and (30) provide in fact strict bounds on these invariants. As $S_2(\rho_{AB}) \geq 0\ \forall\ \rho_{AB}$, Eq. (24) implies

$$|\boldsymbol{r}_A|^2 + |\boldsymbol{r}_B|^2 + ||J||^2 \leq 3\,, \qquad (35)$$

with $|\boldsymbol{r}_A|^2 + |\boldsymbol{r}_B|^2 + ||J||^2 = 3$ if and only if ρ_{AB} is pure ($\rho_{AB}^2 = \rho_{AB}$, $S_2(\rho_{AB}) = 0$). Moreover, as $\mathrm{Tr}\,\rho^{q'} \leq \mathrm{Tr}\,\rho^q$ if $q' > q > 0$, for the present normalization we have $S_3(\rho) \geq \frac{2}{3}S_2(\rho)$, which for a two qubit state implies

$$\boldsymbol{r}_A^t J\boldsymbol{r}_B - \det J \leq 1 - \tfrac{1}{3}S_2(\rho_{AB})\,, \qquad (36)$$

with $\boldsymbol{r}_A^t J\boldsymbol{r}_B - \det J = 1$ if and only if ρ_{AB} is pure. We can verify these results by writing a pure state of two qubits in the Schmidt basis, $|\Psi_{AB}\rangle = \sqrt{p}\,|00\rangle + \sqrt{1-p}\,|11\rangle$, with $p \in [0, 1]$, which leads to $|\boldsymbol{r}_A| = |\boldsymbol{r}_B| = |2p - 1|$, $||J||^2 = 1 + 8p(1-p)$, $\boldsymbol{r}_A^t J\boldsymbol{r}_B = (2p-1)^2$ and $\det J = -4p(1-p)$, and hence to equality in (35)–(36).

An important final remark concerning the quadratic and cubic entropies is that for an arbitrary single qubit state $\rho_A = \frac{1}{2}(I_2 + \boldsymbol{r}_A\cdot\boldsymbol{\sigma})$ they are *identical*, since $\mathrm{tr}\sigma_\mu^m = 0$ for m odd:

$$S_2(\rho_A) = S_3(\rho_A) = 1 - |\boldsymbol{r}_A|^2\,. \qquad (37)$$

This entails that the corresponding entanglement monotones [45] for a two-qubit state are also *identical* [17], coinciding with the square of the concurrence C_{AB} [36, 37]. *Both* quantities I_2^B and I_3^B reduce then to the squared concurrence C_{AB}^2 in the case of a pure two-qubit state.

This last result can be directly verified using the previous Schmidt decomposition: Both matrices M_2 and M_3 become diagonal in the ensuing z basis, their two lowest eigenvalues being identical: $\lambda_2 = \lambda_3 = 4p(1-p) = -\det J$. Eqs. (28) and (31) lead then to $I_2^B = I_3^B = 4p(1-p)$, which is just the square of $C_{AB} = 2\sqrt{p(1-p)}$.

III. APPLICATION

A. States with maximally mixed reduced states

As a first example, let us consider the case $\boldsymbol{r}_A = \boldsymbol{r}_B = \boldsymbol{0}$ in (17), such that $\rho_A = \rho_B = \frac{1}{2}I$ and

$$\rho_{AB} = \tfrac{1}{4}(I + \boldsymbol{\sigma}_A^t J\boldsymbol{\sigma}_B)\,. \qquad (38)$$

We will show that for the state (38):
a) The measurement direction $\boldsymbol{k}$ in system B minimizing I_f^B is *universal*, i.e., the same for any entropy S_f, and given by that of the eigenvector with the largest eigenvalue of the matrix $J^t J$.
b) The ensuing minimum information loss is given by

$$I_f^B(\rho_{AB}) = 2f(\tfrac{p_1+p_2}{2}) + 2f(\tfrac{p_3+p_4}{2}) - f(p_1) - f(p_2) - f(p_3) - f(p_4)\,, \qquad (39)$$

where (p_1, p_2, p_3, p_4) are the eigenvalues of (38) *sorted in decreasing order.*
c) $I_f^A = I_f^B\ \forall\ f$, the minimizing direction in A being that of the eigenvector with the largest eigenvalue of JJ^t.
Proof of a): For $\boldsymbol{r}_A = \boldsymbol{r}_B = \boldsymbol{0}$, the eigenvalues (21) of ρ'_{AB} become $p_\nu^\mu(\boldsymbol{k}) = \frac{1}{4}(1 + \nu\mu|J\boldsymbol{k}|)$, being two-fold degenerate. If $\boldsymbol{k}_m$ is the normalized eigenvector with the largest eigenvalue (J_m^2) of $J^t J$, we have $|J\boldsymbol{k}| = \sqrt{\boldsymbol{k}^t J^t J\boldsymbol{k}} \leq \sqrt{\boldsymbol{k}_m^t J^t J\boldsymbol{k}_m} = |J_m|$ for any unit vector $\boldsymbol{k}$, and hence $p_\mu^\mu(\boldsymbol{k}) \leq p_\mu^\mu(\boldsymbol{k}_m)$. This implies that the distribution $\{p_\mu^\nu(\boldsymbol{k})\}$ is *majorized* [41] by $\{p_\mu^\nu(\boldsymbol{k}_m)\}$, i.e.,

$$\rho'_{AB}(\boldsymbol{k}) \prec \rho'_{AB}(\boldsymbol{k}_m) = \tfrac{1}{4}[I + J_m(\tilde{\boldsymbol{k}}_m\cdot\sigma_A)(\boldsymbol{k}_m\cdot\sigma_B)]\,, \qquad (40)$$

where $\tilde{\boldsymbol{k}}_m = J\boldsymbol{k}_m/J_m$ is the corresponding eigenvector of JJ^t, entailing $S_f(\rho'_{AB}(\boldsymbol{k})) \geq S_f(\rho'_{AB}(\boldsymbol{k}_m))$ and hence $I_f^{\boldsymbol{k}} \geq I_f^{\boldsymbol{k}_m}\ \forall\ \boldsymbol{k}$ and S_f. The state $\rho'_{AB}(\boldsymbol{k}_m)$ is thus the *least mixed* classical state associated with ρ_{AB}, and measurement along $\boldsymbol{k}_m$ the *least disturbing local measurement* (in B) for *any* S_f. Accordingly, the general stationary condition (19) leads in this case to the eigenvalue equation $J^t J\boldsymbol{k} = \lambda\boldsymbol{k}\ \forall\ f$, with both matrices M_2 and M_3 of Eqs. (26), (32) reducing to $J^t J$.

This result is apparent. The local axes can be always chosen such that the matrix J is *diagonal.* This can be achieved through its singular value decomposition $J = U_A J^d U_B^t$, where $J_{\mu\nu}^d = J_\mu\delta_{\mu\nu}$, with J_μ^2 the eigenvalues of $J^t J$ (the same as those of JJ^t) and U_A, U_B orthonormal matrices ($U_\alpha U_\alpha^t = I$). The signs of the J_μ should be chosen such that U_α are rotation matrices ($\det U_\alpha=+1$). Replacing $\boldsymbol{\sigma}_\alpha\to U_\alpha\boldsymbol{\sigma}_\alpha$ in (38), we then obtain

$$\rho_{AB}=\tfrac{1}{4}(I+\sum_{\mu=x,y,z}J_\mu\sigma_{A\mu}\sigma_{B\mu})\,. \quad (41)$$

Since $|J_m|=\text{Max}\{|J_\mu|\}$, the universal least disturbing measurement is, therefore, *along the maximally correlated direction*, leaving the largest term of (41) in the post-measurement state (40). Note that Eq. (41) satisfies Eqs. (15) in a product basis formed by the eigenstates of $\sigma_{A\mu}\sigma_{B\mu}$, for any $\mu=x,y,z$.

Proof of b): Eq. (41) is diagonal in the Bell basis $\{|\Psi_{1,2}\rangle=\frac{|00\rangle\pm|11\rangle}{\sqrt{2}},\ |\Psi_{3,4}\rangle=\frac{|01\rangle\pm|10\rangle}{\sqrt{2}}\}$, i.e., $\rho_{AB}=\sum_i p_i|\Psi_i\rangle\langle\Psi_i|$, with eigenvalues

$$p_{1,2}=\tfrac{1+J_z\pm(J_x-J_y)}{4},\ p_{3,4}=\tfrac{1-J_z\pm(J_x+J_y)}{4}\,.$$

Without loss of generality we may always choose the local axes x,y,z such that $|J_m|=|J_z|\geq|J_x|\geq|J_y|$, with $J_z\geq 0$, $J_x\geq 0$ (rotations of angle π around one of the axes in A or B lead to $J_\mu\to -J_\mu$ for the other axes). In such a case $p_1\geq p_2\geq p_3\geq p_4$, and the least disturbing measurement is along z, such that Eq. (40) becomes

$$\rho'_{AB}(\boldsymbol{k}_m)=\tfrac{1}{4}(I+J_z\sigma_{Az}\sigma_{Bz})\,, \quad (42)$$

having degenerate eigenvalues

$$\tfrac{1+J_z}{4}=\tfrac{p_1+p_2}{2}\,,\quad \tfrac{1-J_z}{4}=\tfrac{p_3+p_4}{2}\,.$$

The minimum information loss $I_f^B(\rho_{AB})=S_f(\rho'_{AB}(\boldsymbol{k}_m))-S_f(\rho_{AB})$ becomes therefore Eq. (39), where (p_1,p_2,p_3,p_4) are in general the eigenvalues of ρ_{AB} sorted in decreasing order.

Proof of c): Since Eq. (39) is fully determined by the sorted eigenvalues of ρ_{AB}, we have obviously $I_f^A=I_f^B$, a result which is apparent from the symmetric representation (41). From (40) it is seen that the minimizing measurement in A is along $\tilde{\boldsymbol{k}}_m$.

Let us now discuss the main features of Eq. (39). It is verified that the strict concavity of f ensures $I_f^B(\rho_{AB})\geq 0\ \forall\ S_f$, with $I_f^B(\rho_{AB})=0$ *only if* $p_1=p_2$ *and* $p_3=p_4$, in which case $\rho_{AB}=\rho'_{AB}=p_1(|00\rangle\langle 00|+|11\rangle\langle 11|)+p_3(|01\rangle\langle 01|+|10\rangle\langle 10|)$ *is a classically correlated state.*

In the von Neumann case $f(p)=-p\log p$, Eq. (39) is just the quantum discord $D^A=D^B$ of the state, coinciding with the result of ref. [29]. For the states (38), $\rho'_B=\rho_B=\frac{1}{2}I$ for any M_B, entailing that the quantum discord (11) reduces to the information deficit, i.e., to the present quantity I_f^B for the von Neumann choice of f.

In the quadratic case (7), Eq. (28) or (39) lead to

$$I_2^B(\rho_{AB})=\tfrac{1}{2}(J_x^2+J_y^2)=(p_1-p_2)^2+(p_3-p_4)^2\,, \quad (43)$$

which is just twice the geometric discord of the state, whereas in the cubic case (30), Eqs. (31) or (39) lead to

$$I_3^B(\rho_{AB})=\tfrac{1}{4}(J_x^2+J_y^2)-\tfrac{1}{2}J_xJ_yJ_z \quad (44)$$
$$=(p_1-p_2)^2(p_1+p_2)+(p_3-p_4)^2(p_3+p_4) \quad (45)$$

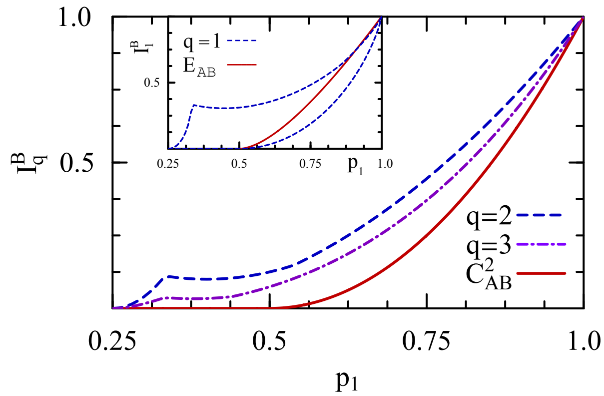

FIG. 1. The maximum and minimum values reached by the quantum correlation measures $I_2^B(\rho_{AB})$ and $I_3^B(\rho_{AB})$ in the state (38), Eqs. (43)–(45), as a function of its maximum eigenvalue p_1. The common minimum is just the squared concurrence C_{AB}^2, whereas the respective maximum is indicated by the dashed and dashed-dotted lines. The inset depicts the maximum and minimum values reached in this state by I_f^B in the von Neumann case ($q=1$, $log=log_2$), where it coincides with the quantum discord, with the solid line depicting the entanglement of formation. The least disturbing measurement is here the same for all entropies, and along the direction of the main principal axis of J^tJ (see text). Quantities plotted are dimensionless in all figures.

which is just the average of the terms in (43) and implies $I_3^B(\rho_{AB})\leq I_2^B(\rho_{AB})$.

Let us notice that for small J_μ, Eq. (39) becomes in fact proportional to (43) *for any* S_f: Setting $J_m=J_z$,

$$I_f^B(\rho_{AB})\approx\tfrac{1}{2}c_f(J_x^2+J_y^2)+O(J^3)=c_fI_2^B(\rho_{AB})+O(J^3) \quad (46)$$

with $c_f=-\frac{1}{4}f''(\frac{1}{4})>0$. This implies *a universal behavior* in the vicinity of the maximally mixed state $I/4$, in agreement with the general results of [17].

Relation with entanglement. It is well known that the state (38) is entangled if and only if its largest eigenvalue p_1 satisfies $p_1>1/2$. Its concurrence [36] is given by

$$C_{AB}=\text{Max}[2p_1-1,0]\,, \quad (47)$$

with $2p_1-1=p_1-p_2-p_3-p_4$. This implies

$$I_2^B\geq C_{AB}^2,\ \ I_3^B\geq C_{AB}^2\,, \quad (48)$$

with equality for $C_{AB}>0$ valid in both cases only if $p_3=p_4=0$ ($C_{AB}^2\leq(p_1-p_2)^2-(p_1-p_2)(p_3+p_4)\leq(p_1-p_2)^2(p_1+p_2)$ if $p_3+p_4\leq p_1-p_2$). Eq. (48) means that for the states (38), I_2^B and I_3^B are both *upper bounds* to their corresponding entanglement monotone. This is not a general property. For instance, it is not valid in the von Neumann case $f(\rho)=-\rho\log\rho$, where Eq. (39) can be lower than the entanglement of formation $E_{AB}=\sum_{\nu=\pm}f(\frac{1+\nu\sqrt{1-C_{AB}^2}}{2})$ [36] for the present states.

Fig. 1 depicts the maximum and minimum values reached by I_2^B and I_3^B in the states (38) for fixed values of the maximum eigenvalue p_1. The common minimum is just the squared concurrence C_{AB}^2, reached for

$p_3 = p_4 = 0$ if $p_1 \geq 1/2$ (and $p_2 = p_1$, $p_3 = p_4$ if $p_1 \leq 1/2$). The maximum is reached for $p_2 = p_3 = p_4$ if $p_1 \geq 7/13 \approx 0.54$ for I_2 and $p_1 \gtrsim 0.44$ for I_3, and for $p_2 = p_3$, $p_4 = 0$, if p_1 lies below the previous values and above $1/3$. As a result, the maximum values for zero concurrence of I_2 and I_3 within these states are $1/8$ and $2/27$ respectively, obtained at $p_1 = 1/2$.

In contrast, in the von Neumann case the minimum (again obtained for $p_3 = p_4 = 0$ if $p_1 \geq 1/2$) lies clearly below E_{AB} $\forall$ $p \in (1/2, 1)$, and even the maximum (attained at $p_2 = p_3 = p_4$ if $p \gtrsim 0.86$ and $p_2 = p_3$, $p_4 = 0$ if $1/3 \leq p_1 \lesssim 0.86$) lies *below* E_{AB} if $p_1 \gtrsim 0.91$. If $p \leq 1/3$ the maximum in these three measures is reached for $p_2 = p_3 = p_1$.

B. States with parity symmetry

Let us now consider the case where both $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ are directed along the same principal axis, i.e., $\boldsymbol{r}_B$ along $\boldsymbol{k}$ and $\boldsymbol{r}_A$ along $J\boldsymbol{k}$, with $\boldsymbol{k}$ an eigenvector of J^tJ (and hence, $J\boldsymbol{k}$ an eigenvector of JJ^t). Choosing these axes as the local z axes, such that $\boldsymbol{r}_A = r_A\boldsymbol{k}_z$, $\boldsymbol{r}_B = r_B\boldsymbol{k}_z$ and $J_{\mu\nu} = J_\mu\delta_{\mu\nu}$, such state can be written as

$$\rho_{AB} = \tfrac{1}{4}(I + r_A\sigma_{Az} + r_B\sigma_{Bz} + \sum_{\mu=x,y,z} J_\mu\sigma_{A\mu}\sigma_{B\mu}) \quad (49)$$
$$= \frac{1}{4}\begin{pmatrix} a_+ & 0 & 0 & \alpha_+ \\ 0 & c_+ & \alpha_- & 0 \\ 0 & \alpha_- & c_- & 0 \\ \alpha_+ & 0 & 0 & a_- \end{pmatrix}, \quad \begin{array}{rcl} a_\pm &=& 1 + J_z \pm (r_A + r_B) \\ c_\pm &=& 1 - J_z \pm (r_A - r_B) \\ \alpha_\pm &=& J_x \mp J_y \end{array}.$$

where the matrix is the representation in the standard basis of $\sigma_{Az}\sigma_{Bz}$ eigenstates. This state commutes with the spin parity [38] $P_z = -\exp[i\pi(\sigma_{Az} + \sigma_{Bz})/2]$. It is also denoted as an X state [32].

We will now show that *a measurement of $\boldsymbol{\sigma}_B$ along any of the principal axes x, y, z will provide a stationary point of $I_f^{\boldsymbol{k}}$ $\forall$ S_f.*

Proof: For a measurement along the z axis ($\boldsymbol{k} = \boldsymbol{k}_z$), i.e., along the axis where ρ_B is diagonal, $J^tJ\boldsymbol{k}_z = J_z^2\boldsymbol{k}_z$, $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ are all along this axis and Eq. (19) is then trivially satisfied $\forall$ α_i. It is a particular case of Eq. (15), which here holds in the standard basis.

For a measurement along the x axis ($\boldsymbol{k} = \boldsymbol{k}_x$), $J^tJ\boldsymbol{k}_x = J_x^2\boldsymbol{k}_x$ while $\boldsymbol{r}_B \cdot \boldsymbol{k}_x = 0$ and $|\boldsymbol{r}_A + \nu J\boldsymbol{k}_x| = \sqrt{r_A^2 + J_x^2}$. Hence $p_\nu^\mu = \frac{1}{4}(1 + \mu|\boldsymbol{r}_A + \nu J\boldsymbol{k}_x|)$ is independent of ν. This leads to $\alpha_1 = \alpha_2 = 0$ in (20), in which case Eq. (19) is again satisfied. For $\boldsymbol{k} = \boldsymbol{k}_y$ the argument is similar. We also remark that these arguments also apply to the quantum discord (11), as $\eta = 0$ in (23) for $\boldsymbol{k} = \boldsymbol{k}_x$ or $\boldsymbol{k}_y$.

While other stationary directions may also exist, the principal axes are strong candidates for minimizing $I_f^{\boldsymbol{k}}$. Typically, the minimum will be attained for measurements along z if $\text{Max}[|J_x|, |J_y|]$ is sufficiently small, while otherwise measurements along x or y will be preferred. A transition between these two regimes will arise as J_x or J_y increases, whose details will depend on the entropic function and may involve intermediate directions $\boldsymbol{k}$.

Writing $\boldsymbol{k} = (\sin\gamma\cos\phi, \sin\gamma\sin\phi, \cos\gamma)$, these intermediate solutions can be found from Eq. (19), which leads here to $\phi = 0$ or $\phi = \pi/2$ (if $|J_x| > |J_y|$ the minimum corresponds to $\phi = 0$ for *any* S_f, as the ensuing distribution majorizes that for $\phi = \pi/2$) and to $\gamma = 0$ or

$$\cos\gamma = \frac{\alpha_1 r_B + \alpha_2 J_z r_A}{\alpha_3(J_x^2 - J_z^2)}\,, \quad (50)$$

where we have assumed $|J_x| > |J_y|$ such that $\phi = 0$. The intermediate solutions $|\gamma| \in (0, \pi/2)$ of (50), if existent, are *degenerate*, as both choices $\pm\gamma$ lead to the same $I_f^{\boldsymbol{k}}$. Just the principal axes solutions are non-degenerate.

The final expression for I_f^B is formally

$$I_f^B(\rho_{AB}) = \sum_{\mu,\nu=\pm} f(p_\nu^\mu) - f(\lambda_\nu^\mu)\,, \quad (51)$$

where $p_\nu^\mu = \frac{1}{4}(1 + \nu r_B k_z + \mu\sqrt{(r_A + \nu J_z k_z)^2 + J_x^2 k_x^2})$ are the eigenvalues (21) of ρ'_{AB} and λ_ν^μ those of ρ_{AB}:

$$\lambda_\nu^\mu = \tfrac{1}{4}[1 + \nu J_z + \mu\sqrt{(r_A + \nu r_B)^2 + (J_x - \nu J_y)^2}]\,. \quad (52)$$

We can verify the previous results in the quadratic and cubic cases. For an X state both matrices M_2 and M_3 (Eqs. (26), (32)) are diagonal in the principal axes basis:

$$M_{2_{\mu\nu}} = \delta_{\mu\nu}(J_\mu^2 + \delta_{\mu z} r_B^2)\,,$$
$$M_{3_{\mu\nu}} = \delta_{\mu\nu}[J_\mu^2 + \delta_{\mu z}(r_B^2 + 2r_B r_A J_z)]\,.$$

Hence, the optimum measurement will be along the axis with the maximum diagonal value and *no intermediate solutions will arise* (for non-degenerate eigenvalues), as opposed to the general case. Assuming $|J_y| < |J_x|$, a "sharp" $z \to x$ transition for the least disturbing measurement will then take place, the x axis preferred for

$$J_x^2 > J_z^2 + r_B^2\,, \quad q = 2\,, \quad (53)$$
$$J_x^2 > J_z^2 + r_B^2 + 2r_B r_A J_z\,, \quad q = 3 \quad (54)$$

in the quadratic and qubic cases respectively, such that

$$I_2^B(\rho_{AB}) = \tfrac{1}{2}\{J_y^2 + \text{Min}[J_x^2, r_B^2 + J_z^2]\}\,, \quad (55)$$
$$I_3^B(\rho_{AB}) = \tfrac{1}{4}\{J_y^2 - 2J_xJ_yJ_z + \text{Min}[J_x^2, r_B^2 + J_z^2 + 2r_A r_B J_z]\}\,. \quad (56)$$

These expressions are in general no longer upper bounds to the squared concurrence, which for these states is $C_{AB} = \frac{1}{2}\text{Max}[|\alpha_+| - \sqrt{c_+c_-}, |\alpha_-| - \sqrt{a_+a_-}, 0]$. Nonetheless, I_2^B remains an upper bound to C_{AB}^2 in the "z phase", as $C_{AB}^2 \leq \frac{1}{4}(J_x \pm J_y)^2 \leq \frac{1}{2}(J_x^2 + J_y^2)$.

C. Mixture of aligned states

As a particular relevant example of Eq. (49), we will consider the mixture of two states with spins aligned along different directions. Choosing the z axis as the bisector, such state can be written as

$$\rho_{AB} = \tfrac{1}{2}(|\theta\theta\rangle\langle\theta\theta| + |-\theta-\theta\rangle\langle-\theta-\theta|) \tag{57}$$

$$= \frac{1}{4}\begin{pmatrix} a_+ & 0 & 0 & c \\ 0 & c & c & 0 \\ 0 & c & c & 0 \\ c & 0 & 0 & a_- \end{pmatrix}, \quad \begin{matrix} a_\pm & = & (1\pm\cos\theta)^2 \\ c & = & \sin^2\theta \end{matrix} \tag{58}$$

which corresponds to $(J_x, J_y, J_z) = (\sin^2\theta, 0, \cos^2\theta)$ and $r_A = r_B = \cos\theta$ in (49). Here

$$|\theta\rangle = \exp[-i\tfrac{\theta}{2}\sigma_y]|0\rangle = \cos\tfrac{\theta}{2}|0\rangle + \sin\tfrac{\theta}{2}|1\rangle \tag{59}$$

is the state with the spin forming an angle θ with the z axis in the x, z plane. The relevance of this state was discussed in [33]. It represents, roughly, the state of a spin pair in the definite parity ground state of a finite n spin ferromagnetic type XY spin chain in a transverse field for $|B| < B_c$, and the *exact* state of any pair at the immediate vicinity of the factorizing field [38] (neglecting small coherence terms $\propto \cos^{n-2}\theta$).

This state is *separable*, i.e., it is a convex mixture of product states [8], and the concurrence C_{AB} accordingly vanishes $\forall\ \theta$. Nonetheless, it has non-zero discord [33] if $\theta \in (0, \pi/2)$. It will then have non-zero values of *any* I_f^B in this interval, with $I_f^B = I_f^A\ \forall\ S_f$ due to the symmetry of the state. For $\theta = 0$ it is obviously a pure product state, while for $\theta = \pi/2$ it is a *classically correlated state*, i.e., diagonal in a *standard* product basis, implying $I_f^B(\theta) \equiv I_f^B(\rho_{AB}(\theta)) = 0$ for $\theta = 0$ or $\theta = \pi/2$ $\forall\ S_f$.

It can be expected that as θ increases, the least disturbing measurement will change from z to x. In the quadratic and cubic cases, the transition is *sharp*. We obtain, according to Eqs. (53)–(56),

$$I_2 = \tfrac{1}{2}\begin{cases} \sin^4\theta & \theta < \theta_{c2} \\ \cos^2\theta + \cos^4\theta & \theta > \theta_{c2} \end{cases}, \tag{60}$$

$$I_3 = \tfrac{1}{4}\begin{cases} \sin^4\theta & \theta < \theta_{c3} \\ \cos^2\theta + 3\cos^4\theta & \theta > \theta_{c3} \end{cases}, \tag{61}$$

where $\cos^2\theta_{c2} = 1/3$ ($\theta_{c2} \approx 0.61\pi/2$) and $\cos^2\theta_{c3} = (\sqrt{17}-3)/4$ ($\theta_{c3} \approx 0.64\pi/2$), with the minimizing measurement changing from z to x for $\theta > \theta_{ci}$. These two quantities exhibit then a cusp-like maximum at $\theta = \theta_{ci}$, i.e. slightly above $\pi/4$, as seen in Fig. 2.

On the other hand, for other entropies a smooth transition from z to the x direction can arise. For instance, in the von Neumann case z is preferred exactly for $\theta \leq \pi/4$, but x is minimum only for $\theta \gtrsim 0.64\pi/2$. In between, the optimum measurement is obtained for an intermediate angle γ, as determined by Eq. (50), which varies continuously from 0 to $\pi/2$, as seen in Fig. 2. This leads to a smooth maximum, located closer to $\pi/4$. In the case of the quantum discord, the minimizing angle is $\gamma = \pi/2\ \forall\ \theta$, exhibiting then *a different behavior* due to the effect of the local term. In this case a relative entropy, rather than a total entropy, is minimized.

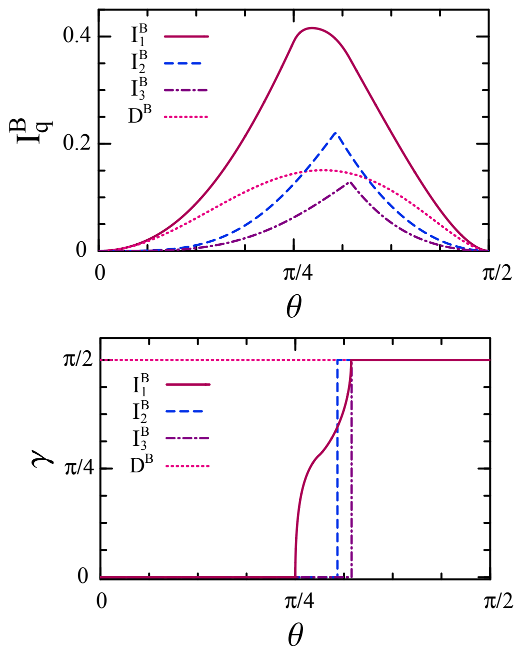

FIG. 2. Top: The quantum correlation measures $I_q^B(\rho_{AB})$ in the state (57), as a function of the angle θ for $q = 1$ (von Neumann case), 2 and 3. D^B denotes the quantum discord. Bottom: The least disturbing measurement angle γ vs. θ for the same cases depicted above. It is seen that γ exhibits a sharp transition from 0 to $\pi/2$ (i.e., from z to the x axis) in the quadratic ($q = 2$) and cubic ($q = 3$) cases, whereas in the von Neumann case ($q = 1$) the transition is smooth. No transition arises in the case of the quantum discord.

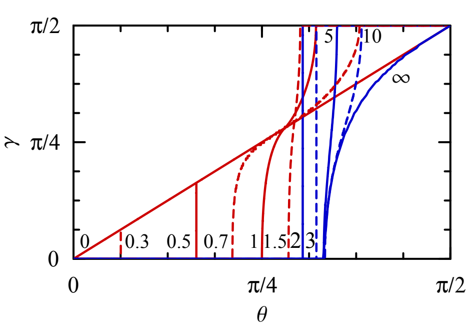

FIG. 3. The least disturbing measurement angle γ vs. θ determined by $I_q^B(\theta)$, for different values of q.

For the present state there is no least mixed state ρ'_{AB}, and the least disturbing measurement depends, therefore, on the entropic function. In order to appreciate previous results from a more general perspective, the behavior of the minimizing angle for different q in the entropies (10) is depicted in Fig. 3. The sharp transition $z \to x$ (i.e., $0 \to \pi/2$) occurs for $2 \leq q \leq 3$, indicating a special critical behavior at these two values. A smoothed transition like that encountered in the von Neumann case arises for $1/2 < q < 2$ and also $q > 3$, where γ varies continuously from 0 to $\pi/2$ within some window of θ values, which narrows for q close to 2 or 3.

For $0 < q \leq 1/2$, the minimizing angle changes sharply from 0 to an intermediate value $\gamma \approx \theta$, increasing then almost linearly with θ ($\gamma \approx \theta$). This is due to the fact that for low q, $S_q(\rho'_{AB})$ is minimized when the lowest eigenvalue of ρ'_{AB} vanishes, and this occurs precisely for $\gamma = \theta$. On the other hand, for high q, $S_q(\rho'_{AB})$ is minimized when the largest eigenvalue of ρ'_{AB} is maximum, and the latter is maximized for $\gamma = 0$ if $\theta \leq \theta_c \approx 0.66\pi/2$, and for an intermediate γ if $\theta > \theta_c$, which varies continuously from 0 to $\pi/2$ for $\theta_c < \theta < \pi/2$. Accordingly, for high but finite q values, $\gamma = 0$ for $\theta \lesssim \theta_c$, increasing then with θ and reaching $\pi/2$ at an increasingly higher θ. Different disorder criteria lead then to different least disturbing measurements in this case, in contrast with the state (38).

IV. CONCLUSIONS

We have analyzed the determination of the minimum information loss I_f^B associated with an unread local measurement in a bipartite system, for a general entropy S_f. Such quantity is a measure of the quantum correlations lost in the local measurement, and reduces to the information deficit and the geometric discord when S_f is chosen as the von Neumann and linear entropy respectively. A general stationary condition was derived, together with its explicit form for an arbitrary mixed state of two qubits. Explicit expressions for the cubic entropy and the associated measure I_3^B were in this case obtained, which require, as in the quadratic case (geometric discord), just the eigenvalues of a 3×3 matrix.

As application, we have first examined two-qubit mixed states with maximally mixed marginals, where the minimum information loss I_f^B for any entropy was shown to be a simple function of the eigenvalues of ρ_{AB}. The minimizing measurement is in this case universal. Moreover, in this case I_2^B and I_3^B were shown to be strict upper bounds of the squared concurrence, which is the associated entanglement monotone for both entropies. We have also analyzed the case of X states, providing explicit expressions for I_2^B and I_3^B and showing that spin measurements along the principal axes of the matrix J^tJ are *universal* stationary points of I_f^B for *any* S_f.

Finally, the special case of a mixture of aligned states was examined in detail. Here the least disturbing local measurement changes, for all measures S_f, from z (bisector axis) to the x axis as the angle 2θ between both directions changes from 0 to π, being then different from that optimizing the original quantum discord (which stays constant), although the type of transition depends on the information measure employed. The least disturbing measurement according to I_f^B is thus more sensible to the strength of the correlation, and reflects the "transition" experienced by the state. Application of the present formalism to more complex systems is currently under investigation.

The authors acknowledge support of CIC (RR) and CONICET (NC,LC) of Argentina.

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