Fractional Fourier transform description with use of differential operators

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The fractional Fourier transform (FRT) is expressed by means of propagation and thin-lens phase delay operators, and a large number of optical systems associated with it are found. At the same time, the output of optical systems is found in terms of the FRT, and the simplicity of the approach is illustrated with two examples. Mathematical definitions for the P-order convolution and correlation are proposed as generalizations of the classical ones such that, when the P-order FRT is applied to them, theorems that generalize the classical convolution and correlation are verified. © 1997 Optical Society of America [S0740-3232(97)01511-1]

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1. INTRODUCTION

The fractional Fourier transform (FRT) and its optical implementation have been the subject of considerable attention in recent times: The concept emerged in quantum mechanics and is a generalization of the Fourier transform (FT) with respect to an order P. Fractional-order convolution and correlation have also been defined and optically generated.[1-14]

The FRT is usually defined through integrals that very much resemble the integral representation of scalar diffraction in the Fresnel region. We present here a description of the same transform, that is, one that is equivalent to it, by using differential operators, namely, the propagation operator and the thin-lens phase delay operator (or function).

The use of operators constitutes a powerful symbolic instrument to describe in a comprehensive way the relevant properties of optical systems without being distracted by many details. Very general properties can be highlighted in this way, and adimensional variables are not required.

Differential exponential operators are recognized to be difficult to treat mathematically,[15-19] and this is indeed so, unless the effect of limiting apertures is not considered. If limiting apertures can be separately treated,[19] considerable insight can be gained by using operators. The advantages of our approach with respect to the abstract use of operators[15-19] are that, although one is required to work with infinite series, the explicit physical representation of every mathematical expression is readily present. This is not so when using abstract operators, such as the scaling operator, which is easily defined through its properties but cannot be straightforwardly implemented optically. When it is necessary to use the scaling operator, we put it in at the final step.

In the appendices there are complementary calculations. There we have found explicitly equivalent systems to one given in Ref. 20. In the main text we use the concept of equivalence to find an infinite number of optical systems giving rise to the FRT, which are equivalent to the simplest one. We have used this concept earlier to associate an infinite set of cylindrical systems with the FT.[21]

We begin in Section 2 by reviewing an auxiliary useful concept, that of the q-index FT.[22] This is not a new concept; it can be recognized as an amplitude- and frequency-scaled version of the ordinary FT.[20] Its introduction makes the following mathematical steps easier.

Different equivalent ways to express the FRT, some of them in terms of operators, are proposed in Section 3. The amplitude distribution on an image is then calculated in terms of the FRT in Section 4. In Section 5 some other mathematical definitions of usual operations are given, namely, P-order convolution and correlation. Both intend to be generalizations of classical definitions such that, when the P-order FRT is applied to them, theorems that generalize the classical convolution and correlation are verified.

2. THEORY REVIEW

We are going to use the bidimensional FT in the form[22]:

$$\mathscr{F}_q\{g\}(x, y) = \exp\left(\frac{j\Delta}{2q}\right)\left\{\exp\left(\frac{-jq}{2}r^2\right) \times \exp\left(\frac{j\Delta}{2q}\right)[g(x, y)]\right\}, \qquad (1)$$

and we are going to call it the q-index ($q > 0$) FT (q-FT), with the operator $\exp(a\Delta)(g) = \Sigma_{n=0}^{\infty}(a^n/n!)\Delta^n(g)$, where $g = g(x, y)$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $r^2 = x^2 + y^2$, and $j = \sqrt{-1}$. We also have, from Eq. (1), $\mathscr{F}_{-q} = \mathscr{F}_q^{-1} = \mathscr{F}_q^{*}$.

When $q = 2\pi$, the classical FT is obtained and for $q > 0$ it coincides with the q-index transform defined by[23]

$$\mathscr{F}_q\{g\}(x, y) = \frac{-jq}{2\pi}\iint_{-\infty}^{\infty} g(u, v) \times \exp[-qj(ux + vy)]\mathrm{d}u\mathrm{d}v, \qquad (2)$$

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where the index q includes a scaling factor. In what follows we are going to work with distributions.[24]

For ease, sometimes we use the following notation to describe optical systems: We call $P_z(g)$ the propagation operator applied to g when it is calculated at a distance z, and L_f the function (or operator) that describes the phase delay in a thin lens with focal distance f (positive or negative). That is,

$$P_z(g) = \exp\left(\frac{jz\Delta}{2k}\right)(g), \tag{3a}$$

$$L_f = \exp\left(\frac{-jk}{2f} r^2\right), \tag{3b}$$

with k the wave number of the light. Equation (3a) is equal to the integral expression[25]

$$P_z(g) = \frac{1}{j\lambda z} \iint g(u, v) \times \exp\left\{\frac{jk}{2z}[(x-u)^2 + (y-v)^2]\right\} \mathrm{d}u\mathrm{d}v = \frac{1}{j\lambda z}\left[g * \exp\left(\frac{jkr^2}{2z}\right)\right]$$

if we take into account Eq. (B3) in Appendix B with $s = k/2z$ and $\lambda = 2\pi/k$.

The effect of the finite size of the pupil of the lens is not considered here in order to keep the results in a closed form.

If we rewrite Eq. (1) with $q = k/f$ and use Eqs. (3), we obtain the identification of the q-FT with the simple system of Fig. 1 and parameter f:

$$\mathscr{F}_q\{g\} = P_f L_f P_f\{g\}. \tag{4}$$

Expression (1) is also equivalent to[22]

$$\mathscr{F}_q\{g\}(x, y) = j \exp\left(\frac{-jqr^2}{2}\right) \times \exp\left(\frac{j\Delta}{2q}\right)\left[\exp\left(\frac{-jqr^2}{2}\right) g(x, y)\right] \tag{5}$$

and, with the use of Eqs. (3) and $q = k/f$, can be written as

$$\mathscr{F}_q\{g\} = L_f P_f L_f\{g\}, \tag{6}$$

which identifies the q-FT with the simple system of Fig. 2 and parameter f.

The general setup shown in Fig. 1 is represented by

$$\bar{g} = P_{z_2} L_f P_{z_1}(g), \tag{7}$$

and the setup shown in Fig. 2 is

$$\bar{g} = L_{f_2} P_z L_{f_1}(g). \tag{8}$$

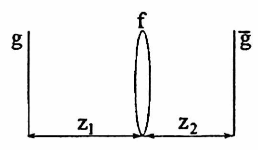

Fig. 1. Optical system to perform the FRT when $z_1 = z_2$.

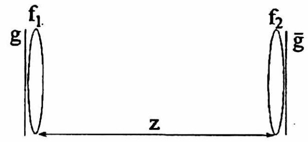

Fig. 2. Optical system to perform the FRT when $f_1 = f_2$.

3. FRACTIONAL TRANSFORM: EQUIVALENT DEFINITIONS

We are going to adopt one expression for the FRT, given by Eq. (9), and we shall find different ways to express Eq. (9), which are given by Eqs. (10)–(16). So we have the following for the FRT of order $P = 2\phi/\pi$ ($\phi \neq m\pi$)[12]:

$$\mathscr{F}^P\{g\}(x, y) = \frac{-jk}{2\pi F \sin\phi} \iint_{-\infty}^{\infty} g(u, v) \times \exp\left[\frac{jk(x^2 + u^2 + y^2 + v^2)}{2F \tan\phi}\right] \times \exp\left[\frac{-jk(xu + yv)}{F \sin\phi}\right] \mathrm{d}u\mathrm{d}v. \tag{9}$$

By using the q-FT with $q = k/F \sin\phi$ and Eq. (2), we can write Eq. (9) as

$$\mathscr{F}^P\{g\}(x, y) = \exp\left(\frac{jkr^2}{2F\tan\phi}\right) \times \mathscr{F}_q\left\{g \exp\left[\frac{jk(u^2 + v^2)}{2F\tan\phi}\right]\right\}(x, y). \tag{10}$$

Equation (10) is used to calculate the FRT from the FT as in Appendix C.1.

In terms of operators [Eq. (1) or (5)], Eq. (10) is

$$\mathscr{F}^P\{g\}(x, y) = \exp\left(\frac{jkr^2}{2F\tan\phi}\right) \times \mathscr{F}_q\left\{g \exp\left(\frac{jkr^2}{2F\tan\phi}\right)\right\}(x, y). \tag{11}$$

Equation (11) can be represented by the system of Fig. 3 with $f_3 = f_4 = -F\tan\phi$. If $\phi = \pi/2$ in Eqs. (9)–(11), then $\mathscr{F}^1 = \mathscr{F}_q$, where $q = k/F$.

The FRT defined by Eq. (9) is valid for $\phi \neq m\pi$, $m = 0, 1, 2, \ldots$. If $\phi = m\pi$ ($P = 2m$), it is defined as

$$\mathscr{F}^{4m}\{g\} = g, \qquad \mathscr{F}^{4m+2}\{g\} = -g^{-},$$

where $g^{-}(x, y) = g(-x, -y)$.

It can be seen from the last result and Eq. (9) that $\mathscr{F}^P$ is periodic with period 4. It can be proved that it is enough to work with $0 < P < 1$, that is, $0 < \phi < \pi/2$.

Using Eqs. (1) and (11) and Eqs. (5) and (11), we obtained, respectively,

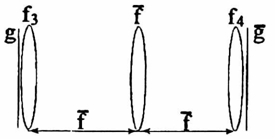

Fig. 3. Optical system equivalent to the systems in Figs. 1 and 2, with the conditions deduced in Appendix A.

$$\mathscr{F}^P\{g\}(x, y) = \exp\left[\frac{jF(1-\cos\phi)\Delta}{2k\sin\phi}\right]\exp\left[\frac{-jk(\sin\phi)r^2}{2F}\right] \times \exp\left[\frac{jF(1-\cos\phi)\Delta}{2k\sin\phi}\right](g), \qquad (12)$$

$$\mathscr{F}^P\{g\}(x, y) = j\exp\left[\frac{-jk(1-\cos\phi)r^2}{2F\sin\phi}\right]\exp\left[\frac{jF(\sin\phi)\Delta}{2k}\right] \left\{g\exp\left[\frac{-jk(1-\cos\phi)r^2}{2F\sin\phi}\right]\right\}. \qquad (13)$$

Equations (12) and (13) represent the two simple systems giving the FRT (Figs. 1 and 2).

Equation (9) can also be written as

$$\mathscr{F}^P\{g\}(x, y) = \frac{-jq}{2\pi}\exp\left(\frac{-jkr^2}{2F\cot\phi}\right)\int\int_{-\infty}^{\infty} g(u, v) \times \exp\left\{\frac{jk}{2F\tan\phi}\left[\left(u - \frac{x}{\cos\phi}\right)^2 + \left(v - \frac{y}{\cos\phi}\right)^2\right]\right\}\mathrm{d}u\mathrm{d}v, \qquad (14)$$

with $q = k/F\sin\phi$.

Equation (14) expresses a convolution. If $h(x, y) = \exp(jkr^2/2F\tan\phi)$, then it can be written as

$$\mathscr{F}^P\{g\}(x, y) = \frac{-jq}{2\pi}\exp\left(\frac{-jkr^2}{2F\cot\phi}\right)(g * h)\left(\frac{x}{\cos\phi}, \frac{y}{\cos\phi}\right). \qquad (15)$$

If we use Eq. (B3) from Appendix B, this last equation becomes

$$\mathscr{F}^P\{g\}(x, y) = \frac{1}{\cos\phi}\left\{\exp\left[\frac{-jk\sin(2\phi)}{4F}r^2\right] \times \exp\left[\frac{jF(\tan\phi)\Delta}{2k}\right][g(x, y)]\right\}\left(\frac{x}{\cos\phi}, \frac{y}{\cos\phi}\right), \qquad (16)$$

with $\phi \neq (2m+1)\pi/2$, $m = 0, 1, 2, \ldots$. Equation (16) describes the FRT as a propagation followed by a lens and a change of scale in the output.

Equations (14)–(16) are not applicable to the ordinary FT, that is, when $\phi = \pi/2$. Equation (16) can be used to define the FRT when $\phi = m\pi$ and shows continuity with the above definition.

An important property of the FRT is order additivity. From Eq. (12) it can be seen that $(\mathscr{F}^P)^{-1} = (\mathscr{F}^P)^* = \mathscr{F}^{-P}$. The inverse of the FRT, $(\mathscr{F}^P)^{-1}$, can also be obtained by changing F to $-F$.

4. OPTICAL SYSTEMS AND THE FRACTIONAL FOURIER TRANSFORM

A. Optical Systems Description with the Fractional Fourier Transform

We are going to describe the systems in Figs. 1 and 2 by means of the FRT. Equation (11) lets us write the q-FT in terms of the P-order FRT (P-FRT):

$$\mathscr{F}_q\{g\}(x, y) = \exp\left(\frac{-jkr^2}{2F\tan\phi}\right)\mathscr{F}^P\left\{g\exp\left(\frac{-jkr^2}{2F\tan\phi}\right)\right\}, \qquad (17)$$

with $q = k/F\sin\phi$.

For the system in Fig. 1, with $1/f \neq 1/z_1 + 1/z_2$ and Eqs. (A1) and (A2) in Appendix A, we have

$$\bar{g} = \exp\left[\frac{-jkr^2}{2A}(z_1 - f)\right]\mathscr{F}_q\left\{g\exp\left[\frac{-jkr^2}{2A}(z_2 - f)\right]\right\}, \qquad (18)$$

with $q = kf/A$, where $A = f(z_1 + z_2) - z_1z_2$.

In Eq. (18) we put $\mathscr{F}_q$ as in Eq. (17) and obtain

$$\bar{g} = \exp\left[\frac{-jkr^2}{2}\left(\frac{z_1 - f}{A} + \frac{1}{F\tan\phi}\right)\right] \times \mathscr{F}^P\left\{g\exp\left[\frac{-jkr^2}{2}\left(\frac{z_2 - f}{A} + \frac{1}{F\tan\phi}\right)\right]\right\}, \qquad (19)$$

with $F\sin\phi = A/f$.

If, in Eq. (19), we take $F\tan\phi = A/(f - z_2)$, we obtain the following for the FRT:

$$\bar{g} = \exp\left[\frac{-jk(z_1 - z_2)r^2}{2A}\right]\mathscr{F}^P\{g\}, \qquad (20a)$$

with

$$\cos\phi = (f - z_2)/f, \qquad F = A/[z_2(2f - z_2)]^{1/2}. \qquad (20b)$$

Equations (20) are not valid, in the real field, for $z_2 \leqslant 0$ and $z_2 \geqslant 2f$. For those values it is convenient to work with the classical FT. But if we want to use the FRT, we can separate the last propagation into two successive propagations $z_2 = z + (z_2 - z)$ such that the distance z satisfies Eqs. (20) and the propagation of distance $(z_2 - z)$ can be expressed by means of Eq. (13).

It can be seen from Eq. (20a) that not all output distributions correspond to a FRT. Besides, when only intensity is measured, so that the phase factor produces no effect, the result is always proportional to a squared FRT of a certain order given by Eqs. (20b).

If in Eq. (19) we take $F\tan\phi = A/(f - z_1)$, we obtain

$$\bar{g} = \mathscr{F}^P\left\{g\exp\left[\frac{-jk(z_2 - z_1)r^2}{2A}\right]\right\}, \qquad (21)$$

with $\cos\phi = (f - z_1)/f$ and $F = A/[z_1(2f - z_1)]^{1/2}$.

Then Eqs. (20) and (21) are descriptions of the optical system in Fig. 1 by means of the FRT.

B. Equivalent Systems Representing the Fractional Fourier Transform

The FRT can be represented by many optical systems (actually, an infinite number of systems), some of which we are going to describe.

From Eqs. (20) we see that the system in Fig. 1, with a lens at the output with focus $\hat{f} = A/(z_2 - z_1)$, represents $\mathscr{F}^P$. It is shown in Fig. 4.

Conversely, given ϕ and F, there are infinite systems associated with $\mathscr{F}^P$ $(P = 2\phi/\pi)$. They are the systems described in Fig. 4, and

$$z_1 = F \tan \phi - f\frac{1 - \cos \phi}{\cos \phi}, \qquad z_2 = f(1 - \cos \phi),$$

$$\hat{f} = fF\frac{\cos \phi}{f \sin \phi - F},$$

with $\phi \neq (2m + 1)\pi/2$.

That is, for each f we have an optical system representing $\mathscr{F}^P$. In general, as the FRT is determined by ϕ and F and the system in Fig. 1 has three free parameters (z_1, z_2, and f), it is possible to choose one of them arbitrarily to represent the FRT. If, for example, we take $z_1 = z_2 = \bar{z}$, then the output lens collapses and Eq. (20a) becomes

$$\mathscr{F}^P\{g\} = \bar{g}, \tag{22}$$

with $\bar{z} = F(1 - \cos \phi)/\sin \phi$ and $f = F/\sin \phi$. Then Eq. (22) coincides with Eq. (12).

Conversely, given the system in Fig. 1, with $z_1 = z_2 = \bar{z}$, the associated FRT has the parameters

$$\cos \phi = (f - \bar{z})/f, \qquad F = A/[\bar{z}(2f - \bar{z})]^{1/2}.$$

As the system in Figs. 1 and 2, with the conditions given in Appendix A, are equivalent, the system in Fig. 2, with a lens of focal length $\bar{f} = f_1 f_2/(f_2 - f_1)$ at the output, represents the FRT. But this means that $f_1 = f_2 = \bar{f}$, so the system in Fig. 2 represents the FRT with

$$\cos \phi = (\bar{f} - z)/\bar{f}, \qquad F = z\bar{f}/[z(2\bar{f} - z)]^{1/2};$$

or, alternatively, the FRT can be represented by the system in Fig. 2 with

$$z = F \sin \phi, \qquad \bar{f} = F \sin \phi/(1 - \cos \phi), \tag{23}$$

which coincides with the FRT description in Eq. (13).

Then, from Fig. 4, we can say that a new expression for the P-FRT using operators is

$$\mathscr{F}^P\{g\}(x, y)$$

$$= \exp\left[\frac{-jk(f \sin \phi - F)}{2fF \cos \phi} r^2\right] \times \exp\left[\frac{jf(1 - \cos \phi)\Delta}{2k}\right]\exp\left(\frac{-jkr^2}{2f}\right) \times \exp\left[\frac{j}{2k}\left(F \tan \phi - f\frac{1 - \cos \phi}{\cos \phi}\right)\Delta\right]\{g\}$$

with any f; this includes as particular cases Eq. (12) ($z_1 = z_2$) and Eq. (13) ($z_1 = 0$).

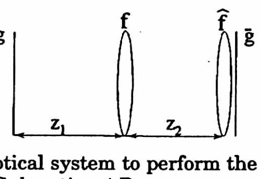

Fig. 4. General optical system to perform the FRT with the parameters given in Subsection 4.B.

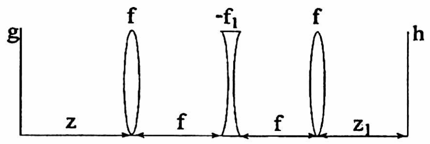

Fig. 5. Optical system described with the FRT in Subsection 4.C.

C. Cascading Systems Description

We can describe the system shown in Fig. 5 as being composed of three FRT cascading systems such as that in Fig. 1; that is, by using Eq. (22) (Ref. 5), we obtain

$$P_{z_1 - z}\mathscr{F}^P \mathscr{F}^{-P_1}\mathscr{F}^P\{g\} = h,$$

with

$$P = 2\phi/\pi, \qquad P_1 = 2\phi_1/\pi, \qquad \cos \phi = 1 - z/f,$$

$$\cos \phi_1 = 1 + (f - z)/f_1, \qquad F = f \sin \phi,$$

$$F_1 = f_1 \sin \phi_1.$$

To be able to apply order additivity, we have to impose the condition that $F = F_1$. This implies that $f_1 = f^2/2(z - f)$, and hence

$$P_{z_1 - z}\mathscr{F}^{2P - P_1}\{g\} = h. \tag{24}$$

In addition, it holds that $2P - P_1 = 2$, and then

$$P_{z_1 - z}\{-g^{-}\} = h, \tag{25}$$

which implies that the complex system in Fig. 5 with the input g has the same behavior as a propagation of distance $(z_1 - z)$ with the input function $-g^{-}$.

If $z_1 = z$ in Eq. (25), we have $h = -g^{-}$, with $f_1 = f^2/2(z - f)$ and f and z arbitrary. Then we can freely take any two parameters (of the four parameters of the system) to obtain $-g^{-}$ as the image through the system in Fig. 5. That is so because in Eq. (24) with $z = z_1$ there is only one FRT.

If we want to find an alternative system that gives, for any input function g, the output $-g^{-}$, we put $\mathscr{F}^{2-P}\mathscr{F}^P\{g\} = -g^{-}$ for any P. This system involves the successive application of the system in Fig. 2 with the parameters given by Eqs. (23) and then the system in Fig. 2 with parameters given by Eq. (23) but with ϕ changed to $\pi - \phi$. This procedure is shown in Fig. 6, where the parameters are

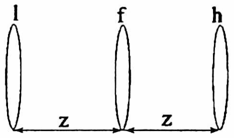

Fig. 6. Optical system to perform $\mathcal{F}^{2-P}\mathcal{F}^{P} = \mathcal{F}^{2}$ with any P, whose elements are analyzed in Subsection 4.C.

$$z = F\sin\phi, \qquad l = F\sin\phi/(1-\cos\phi),$$
$$f = F\sin\phi/2, \qquad h = F\sin\phi/(1+\cos\phi)$$

for any F and $\phi(\neq m\pi)$.

5. CONVOLUTION AND CORRELATION

We propose here definitions of convolution and correlation associated with the FRT. In the first place, it is required that they generalize the usual definitions of convolution and correlation in the following sense: For $\phi = \pi/2$, the usual definitions should be obtained. For example, the P convolution must verify a convolution theorem for $\mathcal{F}^{P}$ so that for $\phi = \pi/2$ the theorem associated with the FT should result:

$$\mathcal{F}_q\{g * h\} = (2\pi j/q)\mathcal{F}_q\{g\}\mathcal{F}_q\{h\}. \tag{26}$$

A. Definition of Convolution

We define the P-order convolution between the functions g and h, by using the notation $g \overset{P}{*} h$, as

$$\left(g \overset{P}{*} h\right)(x, y) = \exp\left(\frac{-jkr^2}{2F\tan\phi}\right)\left[g\exp\left(\frac{jkr^2}{2F\tan\phi}\right) * h \times \exp\left(\frac{jkr^2}{2F\tan\phi}\right)\right](x, y), \tag{27}$$

with $P = 2\phi/\pi$. In this definition it can be recognized that there is an external phase correction and that the main operation is a classical convolution between two complex functions, which are those to be fractionally convolved, each with the same quadratic phase modification. This phase modification depends on the operation fractional order and the parameter F.

The external phase correction does not affect intensity measurements but must be taken into account if the field distribution obtained in fractional convolution is used as input to other processing operations.

The expression of the P convolution with use of the propagation operator is obtained from Eq. (27) and Eq. (B3) of Appendix B with $s = k/2F\tan\phi$:

$$\left(g \overset{P}{*} h\right)(x, y) = \frac{j2\pi F\tan\phi}{k}\exp\left(\frac{-jkr^2}{2F\tan\phi}\right) \times \left\{\exp\left[\frac{jF(\tan\phi)\Delta}{2k}\right]\left[g(x, y)h(\alpha - x, \beta - y) \times \exp\left(\frac{jkr^2}{2F\tan\phi}\right)\right]\right\}_{\alpha=x,\beta=y}. \tag{28}$$

In each particular case, the most convenient parameter, s, is chosen so as to make the calculation easy. Then the expression generally used is

$$\left(g \overset{P}{*} h\right)(x, y) = \frac{j\pi}{s}\exp\left(\frac{-jkr^2}{2F\tan\phi}\right)\exp\left(\frac{j\Delta}{4s}\right)\left(\left\{g(x, y)h(\alpha - x, \beta - y) \times \exp\left[\frac{jk}{2F\tan\phi}(r^2 + \tilde{r}^2)\right]\exp(-js\tilde{r}^2)\right\}\right)_{\alpha=x,\beta=y}, \tag{29a}$$

with $\tilde{r}^2 = (x-\alpha)^2 + (y-\beta)^2$. For example, if $h = \exp(jar^2)$, we choose $s = a + k/(2F\tan\phi) \neq 0$ and obtain, by applying Eq. (13),

$$\left[g \overset{P}{*} \exp(jar^2)\right](x, y) = \frac{j2\pi F\tan\phi}{k + 2aF\tan\phi}\exp\left(\frac{-jkr^2}{2F\tan\phi}\right) \times \exp\left[\frac{jF(\tan\phi)\Delta}{2(k + 2aF\tan\phi)}\right]\left[g(x, y)\exp\left(\frac{jkr^2}{2F\tan\phi}\right)\right]$$
$$= \frac{j2\pi F\tan\phi}{k + 2aF\tan\phi}\exp\left(\frac{-jkr^2}{F\tan\phi}\right)\mathcal{F}^{P_1}\{g\}, \tag{29b}$$

with P_1 determined by the equations given in Subsection 4.B.

Expression (29b) shows that the P convolution between a function g and a quadratic phase delay is the product of a lens and the P_1-FRT of g. This convolution can be optically obtained by using a lens against the input function transmittance, followed by a free propagation and by a lens with focal length opposite to that of the first one.

From another point of view, this system is shift invariant in a wide sense because the output is the P convolution of the input and the impulse response of the system.

Expression (29b) is valid when $P = 1$, in which case the system reduces to a propagation of distance $z = k/2a$.

B. Convolution Theorem

The P-FRT applied to the P convolution verifies that

$$\mathcal{F}^{P}\left\{g \overset{P}{*} h\right\} = j\lambda F(\sin\phi)\exp\left(\frac{-jkr^2}{2F\tan\phi}\right)\mathcal{F}^{P}\{g\}\mathcal{F}^{P}\{h\}, \tag{30}$$

with $\lambda = 2\pi/k$.

To prove this theorem, we start with $\mathcal{F}^{P}$ as defined in Eq. (11), P convolution as defined in Eq. (27), and $q = k/F\sin\phi$:

$$\mathscr{F}^P\left(g \overset{P}{*} h\right) = \exp\left(\frac{jkr^2}{2F\tan\phi}\right)\mathscr{F}_q\left\{\left(g \overset{P}{*} h\right)\exp\left(\frac{jkr^2}{2F\tan\phi}\right)\right\} = \exp\left(\frac{jkr^2}{2F\tan\phi}\right)\mathscr{F}_q\left\{g\exp\left(\frac{jkr^2}{2F\tan\phi}\right) * h\exp\left(\frac{jkr^2}{2F\tan\phi}\right)\right\}.$$

We apply to the last term the classical convolution theorem (26) and again use Eq. (11):

$$\mathscr{F}^P\left\{g \overset{P}{*} h\right\} = \frac{2\pi j}{q}\exp\left(\frac{jkr^2}{2F\tan\phi}\right) \times \mathscr{F}_q\left\{g\exp\left(\frac{jkr^2}{2F\tan\phi}\right)\right\} \times \mathscr{F}_q\left\{h\exp\left(\frac{jkr^2}{2F\tan\phi}\right)\right\} = \frac{2\pi j}{q}\exp\left(\frac{-jkr^2}{2F\tan\phi}\right)\mathscr{F}^P\{g\}\mathscr{F}^P\{h\};$$

thus Eq. (30) is obtained.

C. Definition of Correlation

We define the P-order correlation between the functions g and h, by using the notation $g \overset{P}{\tilde{*}} h$ as

$$g \overset{P}{\tilde{*}} h = \exp\left(\frac{-jkr^2}{2F\tan\phi}\right)\left[g\exp\left(\frac{jkr^2}{2F\tan\phi}\right) * (-h^-)^*\exp\left(\frac{-jkr^2}{2F\tan\phi}\right)\right]. \quad (31)$$

The asterisk as a superscript indicates conjugation.

D. Correlation Theorem

The P-FRT applied to the P correlation verifies that

$$\mathscr{F}^P\left\{g \overset{P}{\tilde{*}} h\right\} = j\lambda F(\sin\phi)\exp\left(\frac{jkr^2}{2F\tan\phi}\right) \times \mathscr{F}^P\{g\}(\mathscr{F}^P\{h\})^*. \quad (32)$$

To prove this theorem, we use the P-FRT defined by Eq. (11) and the convolution theorem (26), with $q = k/F\sin\phi$:

$$\mathscr{F}^P\left\{g \overset{P}{\tilde{*}} h\right\} = \exp\left(\frac{jkr^2}{2F\tan\phi}\right)\mathscr{F}_q\left\{g\exp\left(\frac{jkr^2}{2F\tan\phi}\right) * (-h^-)^*\exp\left(\frac{-jkr^2}{2F\tan\phi}\right)\right\} = \frac{2\pi j}{q}\exp\left(\frac{jkr^2}{2F\tan\phi}\right) \times \mathscr{F}_q\left\{g\exp\left(\frac{jkr^2}{2F\tan\phi}\right)\right\} \times \mathscr{F}_q\left\{(-h^-)^*\exp\left(\frac{-jkr^2}{2F\tan\phi}\right)\right\}. \quad (33)$$

With the q-FT properties

$$\mathscr{F}_q^{\ *} = \mathscr{F}_q^{\ -1}, \qquad \mathscr{F}_q\mathscr{F}_q\{g\} = -g^-,$$

we obtain

$$\left[\mathscr{F}_q\left\{(-h^-)^*\exp\left(\frac{-jkr^2}{2F\tan\phi}\right)\right\}\right]^* = \mathscr{F}_q^{\ -1}\left\{-h^-\exp\left(\frac{jkr^2}{2F\tan\phi}\right)\right\} = \mathscr{F}_q\left\{h\exp\left(\frac{jkr^2}{2F\tan\phi}\right)\right\}.$$

Then

$$\mathscr{F}_q\left\{(-h^-)^*\exp\left(\frac{-jkr^2}{2F\tan\phi}\right)\right\} = \left(\mathscr{F}_q\left\{h\exp\left(\frac{jkr^2}{2F\tan\phi}\right)\right\}\right)^*.$$

With the use of Eq. (11), the last equation becomes

$$\mathscr{F}_q\left\{(-h^-)^*\exp\left(\frac{-jkr^2}{2F\tan\phi}\right)\right\} = \exp\left(\frac{jkr^2}{2F\tan\phi}\right)(\mathscr{F}^P\{h\})^*.$$

With this equation and Eq. (11), Eq. (33) results in Eq. (32).

When $\phi = \pi/2$, this correlation theorem is the classical FT correlation theorem, which is what we wanted.

The optical implementation of the correlation can be performed by taking g as input, using a complex holographic filter in the P-Fourier plane, adding the lens given in Eq. (32), and obtaining the result in the $(4 - P)$-Fourier plane.

E. Autocorrelation Theorem

If, in Eq. (31), we take $g = h$, we obtain the P-order autocorrelation. From the correlation theorem results the autocorrelation theorem:

$$\mathscr{F}^P\left\{h \overset{P}{\tilde{*}} h\right\} = j\lambda F(\sin\phi)\exp\left(\frac{jkr^2}{2F\tan\phi}\right)|\mathscr{F}^P\{h\}|^2. \quad (34)$$

6. CONCLUSIONS

We have used a description of the Fourier transform (FT) named the q-index FT to analyze the fractional Fourier transform (FRT). When it is used, in operator description, abstract scaling operators do not appear in the middle of system descriptions but, if required, appear only in the final results.

The concept of equivalent optical systems was then used to find an infinite set of systems that represent the FRT. That concept, when applied with the use of operators, results in a set of relations among the parameters (focal lengths and propagation distances), which must be satisfied for the systems to be equivalent. The relations between these parameters and the two parameters that determine the FRT, P and F, were found.

These relations simplify the calculations of the output of any system considered as a cascade and composed of any number of lenses. By considering that both the ϕ and F values can be chosen independently, we can use different focal distances for the lenses in the system. Two application examples were given to illustrate the simplicity of the approach.

The output of a simple optical system with three independent parameters was found in terms of the FRT applied to the input. The output intensity distribution of the system always corresponds to that of a certain FRT.

Some calculations on the FRT and the convolution were given in the appendices.

Mathematical definitions of P convolution and P correlation that generalize the usual ones were proposed and expressed in terms of operators. A physical way to generate a certain P convolution in terms of lenses and propagations was described. More general cases require the use of complex filters, as for example, holographic elements.

APPENDIX A: EQUIVALENCE BETWEEN OPTICAL SYSTEMS

It is possible to prove the equivalence between optical systems with the same number of independent geometrical parameters (the number of propagation distances plus the number of focal distances). In the cases shown in Figs. 1–3, these systems present three nonzero parameters, which are, respectively, (z_1, z_2, f), (f_1, f_2, z), and $(f_3, f_4, \bar{f})$.

The equivalence between the systems in Figs. 1 and 2 is given by[20]

$$z = A/f, \qquad f_1 = A/z_2, \qquad f_2 = A/z_1,$$

or, alternatively,

$$z_1 = zf_1/B, \qquad z_2 = zf_2/B, \qquad f = \frac{f_1 f_2}{B},$$

with $A = (z_1 + z_2)f - z_1 z_2 \neq 0$ and $B = f_1 + f_2 - z \neq 0$, A and B being proportional to the defocusing coefficients associated with the systems in Figs. 1 and 2, respectively.

We are going to find the relations between the parameters to obtain the equivalence of the systems in Figs. 1 and 3.

Thus, given the optical system in Fig. 3, we are going to find f_3, f_4, and $\bar{f}$ such that this system is equivalent to that in Fig. 1. This means that, mathematically,

$$\bar{g} = \exp\left(\frac{jz_2\Delta}{2k}\right)\exp\left(\frac{-jkr^2}{2f}\right)\exp\left(\frac{jz_1\Delta}{2k}\right)(g)$$
$$= \exp\left(\frac{-jkr^2}{2f_4}\right)\mathscr{F}_q\left\{\exp\left(\frac{-jkr^2}{2f_3}\right)(g)\right\}, \tag{A1}$$

with $q = k/\bar{f}$.

We shall prove that

$$\bar{f} = \frac{A}{f}, \qquad f_4 = \frac{A}{z_1 - f}, \qquad f_3 = \frac{A}{z_2 - f}. \tag{A2}$$

From Eqs. (A2) we find that

$$f = \frac{\bar{f}f_3 f_4}{f_3 f_4 - \bar{f}^2}, \qquad z_1 = f\left(1 + \frac{\bar{f}}{f_4}\right),$$
$$z_2 = f\left(1 + \frac{\bar{f}}{f_3}\right). \tag{A3}$$

To prove Eqs. (A2), we find the impulse response of the two systems by using the known results[22]

$$\mathscr{F}_q\{\delta(r^2)\} = -jq/2,$$
$$\exp(ja\Delta)[\delta(r^2)] = -\frac{j}{4a}\exp\left(\frac{jr^2}{4a}\right),$$
$$\exp(b\Delta)[\exp(ar^2)] = \frac{1}{1-4ab}\exp\left(\frac{ar^2}{1-4ab}\right),$$
$$4ab \neq 1, \tag{A4}$$

and obtain

$$-\frac{jkf}{2A}\exp\left[\frac{jk}{2A}(f - z_1)r^2\right] = -\frac{jk}{2\bar{f}}\exp\left(-\frac{jkr^2}{2f_4}\right).$$

From this expression it follows that

$$\bar{f} = A/f, \qquad f_4 = A/(z_1 - f),$$

which are the first two equalities in Eqs. (A2).

If we now take $g = 1$ in Eq. (A1) and we use Eq. (A4) and

$$\mathscr{F}_q\{\exp(jar^2)\} = \frac{q}{2a}\exp\left(-\frac{jq^2r^2}{2a}\right),$$

we obtain

$$\frac{f}{f - z_2}\exp\left[-\frac{jkr^2}{2(f - z_2)}\right]$$
$$= -\frac{f_3}{\bar{f}}\exp\left[-\frac{jk}{2f_4\bar{f}^2}(\bar{f}^2 - f_3 f_4)r^2\right].$$

Then it must be true that

$$\frac{f}{f - z_2} = -\frac{f_3}{\bar{f}},$$

and so the third equality in Eqs. (A2) is obtained.

Furthermore, the systems in Figs. 2 and 3 are also equivalent, with

$$\bar{f} = z, \qquad f_3 = \frac{zf_1}{z - f_1}, \qquad f_4 = \frac{zf_2}{z - f_2},$$

or, alternatively,

$$z = \bar{f}, \qquad f_1 = \frac{f_3\bar{f}}{f_3 + \bar{f}}, \qquad f_2 = \frac{f_4\bar{f}}{f_4 + \bar{f}}.$$

APPENDIX B: CONVOLUTION DEFINED BY MEANS OF THE PROPAGATION OPERATOR

To obtain the convolution of two functions expressed by means of the propagation operator, we use Eq. (2) for $\mathscr{F}_s$ and start with the identity

$$(g * h)(x, y) = \frac{j\pi}{s}\mathscr{F}_s^{-1}\left\{\exp\left[\frac{-js}{4}(\xi^2 + \eta^2)\right] \times \mathscr{F}_s\{g(u, v)h(x - u, y - v) \times \exp(-js\bar{r}^2)\}(\xi, \eta)\right\}(x, y), \quad \text{(B1)}$$

with $\bar{r}^2 = (x - u)^2 + (y - v)^2$ and any s. In this equation we insert the $\mathscr{F}_s$ given by Eq. (1), where only the variables x and y appear, and we obtain

$$(g * h)(x, y) = \frac{j\pi}{s}\left(\mathscr{F}_s^{-1}\left\{\exp\left[\frac{-js}{4}r^2\right] \times \mathscr{F}_s\{g(x, y)h(\alpha - x, \beta - y) \times \exp(-js\bar{r}^2)\}\right\}\right)_{\alpha=x,\beta=y}, \quad \text{(B2)}$$

with $\bar{r}^2 = (\alpha - x)^2 + (\beta - y)^2$.

If we use the FT property

$$\mathscr{F}_s\{g \exp(-js^2ar^2)\} = \exp(ja\Delta)\mathscr{F}_s\{g\},$$

the last equation is

$$(g * h)(x, y) = \frac{j\pi}{s}\left\{\exp\left(\frac{j\Delta}{4s}\right) \times [g(x, y)h(\alpha - x, \beta - y) \times \exp(-js\bar{r}^2)]\right\}_{\alpha=x,\beta=y}. \quad \text{(B3)}$$

Observe that it is possible to calculate improper integrals by performing differentiation only. From Eq. (B3) and $h = 1$, we obtain

$$\int\int_{-\infty}^{\infty} g(x, y)\mathrm{d}x\mathrm{d}y = \frac{j\pi}{s}\left\{\exp\left(\frac{j\Delta}{4s}\right)[g(x, y)\exp(-js\bar{r}^2)]\right\}_{\alpha=x,\beta=y}$$
$$= \frac{j\pi}{s}\left\{\exp\left(\frac{j\Delta}{4s}\right)[g(\alpha - x, \beta - y) \times \exp(-js\bar{r}^2)]\right\}_{\alpha=x,\beta=y}.$$

APPENDIX C: APPLICATIONS

We give here some calculations to show how our approach gives some results in a very simple way.

1. Fractional Fourier Transform of Some Usual Functions

Using Eq. (11) or (16) and q-FT properties,[22] we obtain

$$\mathscr{F}^P\{\exp(jar^2)\} = \frac{k}{2aF\sin\phi + k\cos\phi} \times \exp\left[\frac{jk(2aF\cos\phi - k\sin\phi)r^2}{2F(2aF\sin\phi + k\cos\phi)}\right],$$

with $2aF\sin\phi + k\cos\phi \neq 0$ and any a, (taking $a = 0$, we obtain $\mathscr{F}^P\{1\}$);

$$\mathscr{F}^P\left\{\exp\left(\frac{-jkr^2}{2F\tan\phi}\right)\right\} = \frac{-j2\pi F\sin\phi}{k}\delta(x, y);$$

$$\mathscr{F}^P\{\delta(x, y)\} = \frac{-jk}{2\pi F\sin\phi}\exp\left(\frac{jkr^2}{2F\tan\phi}\right);$$

$$\mathscr{F}^P\{\delta(r - a)\} = -jqaJ_0(qra)\exp\left[\frac{jk(r^2 + a^2)}{2F\tan\phi}\right],$$

with $q = k/F\sin\phi$;

$$\mathscr{F}^P\{J_0(br)\} = \frac{1}{\cos\phi}J_0\left(\frac{br}{\cos\phi}\right) \times \exp\left[\frac{-j(\tan\phi)(k^2r^2 + F^2b^2)}{2Fk}\right].$$

2. P Convolution of Some Usual Functions

From properties of convolution and Eq. (27), we obtain

$$g \overset{P}{*} \delta(x, y) = g(x, y) \qquad \text{for any } P;$$

$$\sum_{-\infty}^{\infty}\delta(x - n, y - m) \overset{P}{*} g(x, y) = \sum_{-\infty}^{\infty} g(x - n, y - m) \times \exp\left\{\frac{-jk}{2F\tan\phi}[n(2x - n) + m(2y - m)]\right\},$$

$$\exp(-ar^2) \overset{P}{*} \exp(-br^2) = \frac{j\pi F\tan\phi}{k + jF(\tan\phi)(a + b)}\exp\left(\frac{-jkr^2}{2F\tan\phi}\right) \times \exp\left\{\frac{(k + j2bF\tan\phi)(jk - 2aF\tan\phi)r^2}{4F(\tan\phi)[k + jF(\tan\phi)(a + b)]}\right\},$$

with $k + jF(\tan\phi)(a + b) \neq 0$.

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