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The parity function in optical waves propagation

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Abstract We propose a generalization of the concept of parity as a continuous function and show that it is conserved in optical waves free propagation. Some properties are commented and an experimental implementation to optically generate a representation is suggested.

Key words: Wave propagation – convolution – parity

1. Introduction

Invariant quantities are important powerful tools in many fields of science because they permit to obtain information about the evolution and behavior of systems. As an example, in physics, is well known the utility of momentum, mass, energy, charge, etc., conservation laws. In general, a conservation law is a statement that establishes that a physical magnitude is unchanged (conserved) in an interaction occurring within a closed system. Parity (from Latin *paritas*: equal or equivalent) is a concept appearing in several sciences and even in several branches of physics with related meanings. In mathematics, it designs the property shared by numbers or functions that are both either even or odd. In computer technology the name indicates whether the number of binary ones in a word is odd or even and is usually used for errors detection. In Quantum Mechanics it is the property of a wave function that describes the behavior of a system whose physical coordinates are related by inversion about a center. If parity is even, the wave function is unchanged and if it is odd, the wave function is changed only in sign. Other examples can be found in crystallography, etc. As can be seen, the concept of parity usually assigns a discrete value 1 or –1 to the magnitude it acts upon.

In the usual mathematics definition of parity there is no intermediate state between an even function and an odd function or a function not exhibiting any obvious parity property. Following a similar scope as that that lead Zernike to propose the concept of *degree of coherence*, it would be very convenient if a continuous magnitude related to the concept of parity could be available. Even more if this magnitude could be measured and predicted under current optical situations. Besides, it is an intuitive idea (that can also be demonstrated) that this concept of parity should be conserved by free propagation. If a continuous generalization of the parity can be implemented its conservation could also be expected.

The aim of this paper is to define a more general concept related to parity, named the parity function, applied to optical fields and to demonstrate its conservation under wave field propagation. In section 2, we define the parity function, some properties are established in section 3, and some examples are shown. In section 4 we present an optical setup to obtain the parity function of an optical field.

2. Definition of the parity function

Let us consider a complex scalar field described by a bounded function $f(x)$, where x is the spatial coordinate. For simplicity we are going to consider the spatial dependence to be one-dimensional. The function can be split in its even and odd parts with respect to the coordinate origin as

$$f(x) = f_P(x) + f_I(x)\,, \tag{1}$$

where the even and odd parts are given by

$$f_P(x) = \frac{f(x) + f(-x)}{2}\,, \tag{2}$$

$$f_I(x) = \frac{f(x) - f(-x)}{2}\,. \tag{3}$$

The point about which the even and odd parts are defined can be generalized.

Even and odd parts with respect to any arbitrary point x_0 can be written as

$$f_P(x; x_0) = \frac{f(x_0 + x) + f(x_0 - x)}{2}\,, \tag{4}$$

$$f_I(x; x_0) = \frac{f(x_0 + x) - f(x_0 - x)}{2}\,. \tag{5}$$

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The parity (P) of the field $f(x)$ in the point x_0 is defined as

$$P\{f\}(x_0) = \frac{\int |f_P(x;x_0)|^2\,\mathrm{d}x}{\int |f(x)||^2\,\mathrm{d}x}, \tag{6}$$

where the integration interval is the support of $f(x)$ and $0 < \int |f(x)|^2\,\mathrm{d}x < \infty$.

Notice that the denominator is a constant value. So the parity in x_0 is determined by the fraction of total energy contained in the even part of the function.

Eq. (6) can be written, by using eq. (4) and the convolution operator [1], indicated as $\otimes$,

$$P\{f\}(x_0) = \frac{1}{2}\left\{1 + \frac{(f \otimes f^*)(2x_0)}{\int |f(x)|^2\,\mathrm{d}x}\right\}, \tag{7}$$

where * indicates complex conjugation.

After eq. (7), but also from its definition, it can be seen that if $f(x)$ has compact support, then the parity consists of a constant additive term $\frac{1}{2}$ plus a bounded function with compact support. As most pupil functions in optics have compact support, the parity of a plane wave after incidence upon that pupil inherits this property. The parity is then a function generally smoother than (or, eventually as smooth as) the pupil function itself.

For eventual cases in which one (or both) of the integrals in (7) is divergent, the parity is defined

$$P\{f\}(x_0) = \frac{1}{2}\left\{1 + \lim_{A\to\infty} \frac{\int_{-A}^{A} f(x_0+x)\,f^*(x_0-x)\,\mathrm{d}x}{\int_{-A}^{A} |f(x)|^2\,\mathrm{d}x}\right\}, \tag{8}$$

with $\int |f(x)|^2\,\mathrm{d}x \neq 0$.

If both integrals in eq. (8) converge, then eq. (8) and eq. (7) coincide.

Examples

From eq. (7) the following parity functions are obtained

i) $f(x) = \begin{cases} (x+\mathrm{i}), & |x| < 1 \\ 0, & |x| > 1 \end{cases}$,

$$P\{f\}(x_0) = \begin{cases} \frac{1}{4}(3 - |x_0|^3), & |x_0| < 1 \\ \frac{1}{2}, & |x_0| > 1 \end{cases}.$$

ii) $f(x) = \begin{cases} \sin^2 x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$,

$$P\{f\}(x_0) = \frac{1}{2}\begin{cases} 1 + \frac{1}{3\pi}\left[(\pi - |x_0|)(2 + \cos 4x_0) + \frac{3}{4}|\sin 4x_0|\right], & |x_0| < \pi \\ 1, & |x_0| > \pi \end{cases}.$$

iii) If $a = a_1 + \mathrm{i}a_2$, $a_1 > 0$,

$$P\{\exp(-ax^2)\}(x_0) = \frac{1}{2}\left(1 + \exp\left(-2\frac{|a|^2}{a_1}x_0^2\right)\right).$$

From eq. (8) the following parity functions are obtained

iv) If $n = 0, 1, 2, \ldots$, $P\{x^n\}(x_0) = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$.

v) If $\mathrm{Re}\, a \neq 0$, $P\{\exp ax\}(x_0) = \frac{1}{2}$.

vi) $P\{\sin x\}(x_0) = \sin^2 x_0$.

vii) $P\{\cos x\}(x_0) = \cos^2(x_0)$.

3. Properties of the parity function

Property 1. The parity function does not change under complex conjugation, as can be seen from eq. (6),

$$P\{f^*\}(x_0) = P\{f\}(x_0). \tag{9}$$

Property 2. The parity function does not change under multiplication of the function f by a complex constant, as can be seen from eq. (6). It is, if $c \in C$,

$$P\{f\}(x_0) = P\{cf\}(x_0), \tag{10}$$

(and then, under general arbitrary phase changes $\exp[\mathrm{i}\varphi]$ with φ a constant).

Property 3. If the imparity function I, of the scalar field f in a point x_0 is defined as

$$I\{f\}(x_0) = \frac{\int |f_I(x, x_0)|^2\,\mathrm{d}x}{\int |f(x)|^2\,\mathrm{d}x}, \tag{11}$$

then it is verified that (see Appendix)

$$P\{f\}(x_0) + I\{f\}(x_0) = 1. \tag{12}$$

Property 4. As a consequence of eq. (12), the parity is bounded to the [0,1] interval, it is $P\{f\}(x_0) \leq 1$.

Property 5. If $f(x)$ is even or odd with respect to $x = 0$, then $P\{f\}(x_0)$ is even with respect to 0, as can be seen from eqs. (4) and (6).

Property 6. If $f(x)$ is even with respect to 0, then $P\{f\}(0) = 1$. If $f(x)$ is odd with respect to 0, then $P\{f\}(0) = 0$. So, if an even function has constant parity then its parity is 1, and if an odd function has constant parity then its parity is 0 (see Appendix).

Property 7. For all $a \in \boldsymbol{R}$,

$$P\{f(ax)\}(x_0) = P\{f(x)\}(ax_0), \tag{13}$$

(see Appendix).

Property 8. If f is such that $\int |f|^2\,\mathrm{d}x < \infty$, then $P\{P\{f\}\}(z_0) = 1$. If $\int |f|^2\,\mathrm{d}x = \infty$, in general it is $P\{P\{f\}\}(z_0) \neq 1$, as the case, $P\{\sin x\}(x_0) = \sin^2 x_0$ and $P\{\sin^2 x_0\}(z_0) = \frac{1}{6}(5 + \cos 4z_0))$, shows (see Appendix).

Property 9. If we assume f with $\int |f|^2 \, dx > 0$, then if $P\{f\}(x_0) = $ constant, f has not bounded support (see Appendix).

Examples

viii) From example v), with $\mathrm{Re}\, a \neq 0$, and eq. (12), it verifies $P\{f\} = I\{f\}$.

ix) From examples vi) and vii), and eq. (12) it can be deduced that $P\{\sin x\} = I\{\cos x\}$.

4. Conservation of the parity

Let us consider the free propagation of scalar fields $f(x)$ with $\int |f|^2 \, dx = 1$, along the (mean) direction of the coordinate axis z, and the p-index Fourier Transform of f [2]

$$\Im_p\{f\}(x) = \sqrt{\frac{p}{2\pi}} \int_{-\infty}^{\infty} f(\xi) \exp(-\mathrm{i}p\xi x) \, d\xi = F_p(x) . \tag{14}$$

Theorem

The necessary and sufficient condition for f and g to have the same parity is

$$F_p(x) \, F_p^*(-x) = G_p(x) \, G_p^*(-x) , \tag{15}$$

assuming that the Fourier Transform of f and g exist.

From eq. (7) it follows that the necessary and sufficient condition for f and g to have the same parity is $f \otimes f^* = g \otimes g^*$. Then, by applying the Fourier Transform, eq. (15) is obtained.

Corollary 1 (Conservation of the parity)

If the function g is the function describing the field f after propagation a distance z, then

$$P\{g\}(x_0) = P\{f\}(x_0) . \tag{16}$$

By using the fact that $G_p(x)$ is related to $F_p(x)$ by [3]

$$G_p(x) = \begin{cases} \exp\left(\mathrm{i}kz\sqrt{1 - \left(\frac{px}{k}\right)^2}\right) F_p(x), & \left|\frac{px}{k}\right| < 1 \\ 0, & \left|\frac{px}{k}\right| > 1 \end{cases} , \tag{17}$$

then eq. (15) is verified. It is the functions g and f have the same parity.

Physically, the conservation of the parity function is a consequence of the conservation of energy and indicates that the energy in the even and odd parts of the function is independently conserved after free propagation. Interference between these parts contains no energy.

An example is the plane wave, showing $P = 1$ for every point x_0. This value is trivially conserved with propagation. Other non-diffracting beams are also trivial examples of the parity conservation as, by definition the fields themselves are conserved.

Corollary 2

Assuming both f and g are odd or both are even, the necessary and sufficient condition for f and g to have the same parity is

$$|F_p(x)| = |G_p(x)| .$$

This follows from eq. (15).

Notice that if f is real, (imaginary), by knowing its parity we can get f from

$$\Im_p\{(2P\{f\} - 1)(x_0)\}(2x) = \tfrac{1}{2} \, F_p(x) \, F_p^*(-x) ,$$

because the second term is $\frac{1}{2} F_p^2(x)$, $(-\frac{1}{2} F_p^2(x))$.

These definitions and results can be easily generalized to bi-dimensional fields.

5. Optical generation of the parity function

The first step to obtain the parity function of an optical field is the making of a Vander Lugt filter matched to it. It can be done by means of a holographic register or by using a spatial light modulator. Figure 1a shows the set up of a typical Vander Lugt filter [3–5].

If we want the representation of the parity of a function $f(x_0, y_0)$, then the Fourier Transform of the field $f(x_0, y_0)$ is made to form on a holographic plate H, where a coherent reference plane wave is also present, and is given by

$$R(x_1, y_1) = R_0 \exp(-2\pi \mathrm{i} y_1 \alpha) , \tag{18}$$

with $\alpha = \dfrac{\sin\theta}{\lambda}$, θ the angle formed by the reference beam with the optical axis and λ the wavelength of the employed light. The transmittance of the filter H, after development with the usual linearity conditions, can be expressed as

$$I(x_1, y_1) = \left| \frac{1}{\lambda d} \, F(x_1/\lambda d, y_1/\lambda d) + R_0 \exp(-2\pi \mathrm{i} y_1 \alpha) \right|^2 , \tag{19}$$

where $F(x_1/\lambda d, y_1/\lambda d)$ is the Fourier Transform of the field $f(x_0, y_0)$ performed by the lens L with focal distance d, and where the proportionality constant due to the development process has been omitted for simplicity. Developing the square of the modulus in eq. (19)

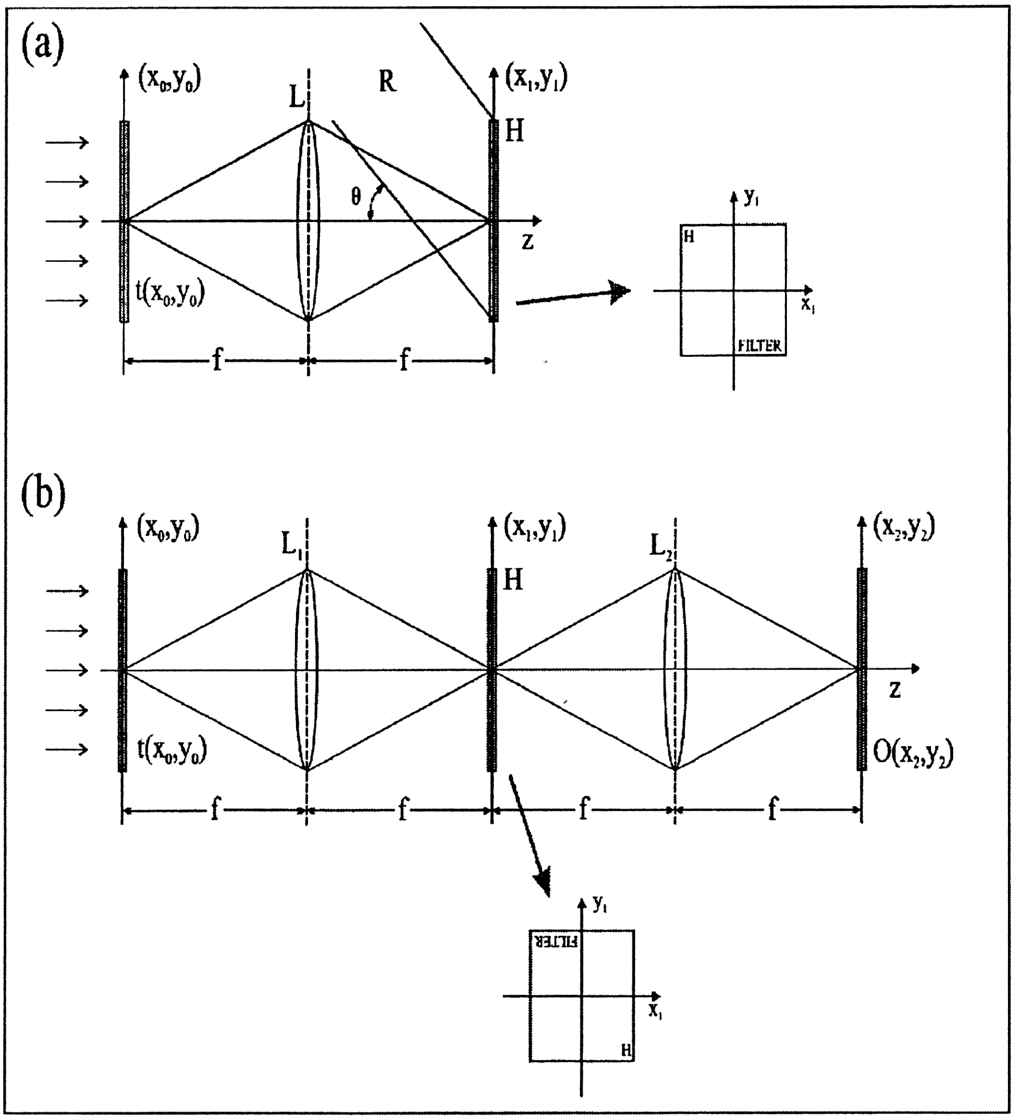

Fig. 1. a) The set-up for the filter construction; b) the generation of the parity function using the matched filter.

we obtain for the filter transmittance,

$$T(x_1,y_1) = R_0^2 + \frac{1}{(\lambda d)^2}\,|F(x_1/\lambda d,\ y_1/\lambda d)|^2 + \frac{R_0}{\lambda d}\,F(x_1/\lambda d,\ y_1/\lambda d)\exp(2\pi i y_1\alpha) + \frac{R_0}{\lambda d}\,F^*(x_1/\lambda d,\ y_1/\lambda d)\exp(-2\pi i y_1\alpha)\,. \tag{20}$$

The second step for obtaining an optical version of the parity function is shown in fig. 1b. The filter H is located in the Fourier plane of a 4-d optical processor after an 180° rotation with respect to the coordinates origin. In this way the transmittance function of the filter can be described as

$$T'(x_1,y_1) = T(-x_1/\lambda f,\ -y_1/\lambda f)\,. \tag{21}$$

If the incoming field to the processor is $f(x_0,y_0)$, then the field immediately behind the filter H due to the first lens L_1, is

$$\frac{1}{\lambda d}\,F(x_1/\lambda d,\ y_1/\lambda d)\,T'(x_1,y_1)\,. \tag{22}$$

The second lens L_2 performs the optical Fourier transform of the field represented by the last expression and multiplies by a $1/\lambda d$ factor, so that the field in the output plane of the processor can be expressed as

$$O(x_2,y_2) = R_0^2 f(-x_2,-y_2) + \frac{1}{(\lambda d)^2}\,f(x_2,y_2)\otimes f^*(-x_2,-y_2)\otimes f(-x_2,-y_2) + \frac{R_0}{\lambda d}\,f(-x_2,-y_2)\otimes f(x_2,y_2)\otimes\delta(x_2,y_2+\alpha\lambda d) + \frac{R_0}{\lambda d}\,f(-x_2,-y_2)\otimes f^*(-x_2,-y_2)\otimes\delta(x_2,y_2-\alpha\lambda d)\,. \tag{23}$$

The last term in eq. (23) represents, but for a bias term and a multiplicative constant, the parity function of the field $f(x_0, y_0)$ in an inverted coordinates system. Then, if α, the reference wave carrier frequency was chosen high enough so that the last term does not spatially overlap with the others, an optical representation of the parity can be obtained.

6. Conclusions

We have defined and proposed the use of the parity function. We have shown several mathematical properties and its conservation under free propagation in optical fields.

When propagation is not free, i.e. when light interacts with pupils, the value of the parity function is not conserved. There is not a simple expression relating the parity of pupils and the parity of the fields before and after interacting with them.

An optical setup to optically generate a representation using a modified version of the classical Vander Lugt filter was also proposed. The concept can be applied as a tool for help in inverse problems.

Also, by taking as hint the use of parity in computation technology, the variation in the parity function where its conservation is expected can be used as a tool for inducing the presence of alterations in the assumed conditions.

The parity function is only one of a set of magnitudes that can be defined in connection with propagation. A closely connected one, which is the hermiticity function, is under development and will be published elsewhere.

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Appendix

Property 3

The left hand term in eq. (12) can be written (by using eqs. (6) and (11)) as

$$\frac{\int |f_P(x;x_0)|^2\,dx + \int |f_I(x;x_0)|^2\,dx}{\int |f(x)|^2\,dx}. \tag{A1}$$

By using eqs. (4) and (5), and taking into account that

$$\int f_P(x;x_0)\, f_I^*(x;x_0)\,dx = 0, \tag{A2}$$

due to the orthogonality between functions $f_P(x;x_0)$ and $f_I^*(x;x_0)$, eq. (A1) equals unity.

Property 6

From definition of convolution is

$$(f\otimes f^*)\,(0) = \int f(x)\, f^*(-x)\,dx = \begin{cases} \int |f(x)|^2\,dx, & f \text{ even} \\ -\int |f(x)|^2\,dx, & f \text{ odd} \end{cases}.$$

From eq. (7) it results

$$P\{f\}\,(0) = \begin{cases} 1, & f \text{ even} \\ 0, & f \text{ odd} \end{cases}. \tag{A3}$$

Property 7

From

$$\int_{-A}^{A} |f(ax)|^2\,dx = \frac{1}{|a|}\int_{-aA}^{aA} |f(x)|^2\,dx, \tag{A4}$$

and

$$(f(ax)\otimes f^*(ax))\,(2x_0) = \int f(ax)\, f^*(a(2x_0 - x))\,dx = \frac{1}{|a|}\int f(\xi)\, f^*(2ax_0 - \xi)\,d\xi,$$

with $\xi = ax$, eq. (13) results.

Property 8

From eq. (6)

$$P\{f\}\,(x_0) = \frac{1}{2}\left\{1 + \frac{(f\otimes f^*)\,(2x_0)}{\int |f(x)|^2\,dx}\right\},$$

if we call

$$h(x_0) = \frac{(f\otimes f^*)\,(2x_0)}{\int |f(x)|^2\,dx},$$

then

$$P\{P\{f\}\}\,(z_0) = \frac{1}{2}\left\{1 + \frac{[1+h(x_0)]\otimes[1+h(x_0)]\,(2z_0)}{\int [1+h(x_0)]^2\,dx_0}\right\}. \tag{A5}$$

The fact that $\int |f(x)|^2\,dx < \infty$, implies $\lim h(x_0)_{x_0\to\pm\infty} = 0$, and then $\int h(x_0)\,dx_0$, $\int h^2(x_0)\,dx_0 < \infty$. By using the definition in eq. (6), it is obtained

$$\lim_{A\to\infty} \frac{2A + \int_{-A}^{A} [2h(\xi) + h(\xi)\,h(2z_0 - \xi)]\,d\xi}{2A + \int_{-A}^{A} [h^2(\xi) + 2h(\xi)]\,d\xi} = 1. \tag{A6}$$

Property 9

If we assume that the support of f is bounded, then $0 < \int |f|^2\,dx < \infty$, and there exist at least x, y such that $(f(x)\otimes f^*(x))(x) = 0$, and $(f(x)\otimes f^*(x))\,(y) \neq 0$, then $P\{f\}$ is not constant.

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